ABSTRACT. A widespread belief is that first-order formal predicate logic can be applied directly to real phenomena. I prove that this is a misconception. Formal logic is only applicable to semantic models (whether of reality or not). Some consequences of this insight are drawn. Problems concerned with existence and reference, which caused for instance Russell and Quine considerable troubles, evaporate easily. Quine's ideas on ontological commitment are untenable. Tarski's definition of truth-in-a-model for formal languages cannot be extended to an acceptable correspondence theory of truth for natural, non-formal languages.

1. Logic, Reality, and Models

1.1 A Misleading Picture. It is commonly believed that formal logic can be applied directly to reality which gives the following picture:

\begin{center}
Formal logic \quad \longrightarrow \quad \text{Reality}
\end{center}

This picture is, however, misleading and gives rise to severe problems.

1.2 The Correct Picture. By a semantic set-theoretic model, we here understand a set theoretical structure of the kind used in the standard semantics for classical first-order predicate logic. The correct picture of applications of logic to problems on reality is:

\begin{center}
Formal logic \quad \longrightarrow \quad \text{Semantic model} \quad \longrightarrow \quad \text{Reality}
\end{center}

Formal logic applies only to semantic models of reality and not directly to reality itself. On the picture 1.1, assuming \( \exists x \, P(x) \) in a set of premises in a formal language implies assuming that there is something in reality having the property \( P \). On the picture 1.2, assuming \( \exists x \, P(x) \) in a set of premises in a formal language implies only that in every model of the set of premises there is something belonging to the set \( P \) representing \( P \) in the model. Nothing is implied about existence in reality. I now show that the second picture is the correct one.
1.3 OBSERVATION. First-order formal predicate logic is applicable to semantic models and only to semantic models and not directly to reality.

PROOF:

It is trivial that formal logic is applicable to models. The existence of this relation is shown in the chapter on semantics in most textbooks of first-order elementary predicate logic, for instance in chapters 8 and 9 in Hansen (2003). To show that formal logic is not applicable directly to reality, we consider an example from physics. Let S be a system of the kind studied in physics, for instance in classical mechanics. Suppose classical formal logic applies to S. After having studied, by the laws of mechanics, how the system S functions, we can make statements about how S reacts under given conditions. Such statements can be expressed in conditionals of the form 'if A then B' where A represents the conditions and B expresses the reaction of the system. Such a conditional is in formal logic represented by 'A → B' where '→' is used to represent the non-formal operator 'if-then'. By the hypothesis, 'A → B' represents something in S itself. In the semantics for formal classical logic, the truth condition for A → B is based on a truth-functional connection which exists in the models.

More precisely, from the truth function for A → B, we can, as in Hansen (1996b), derive the truth condition:

\[ (3-1) \quad A \rightarrow B \text{ is true } \iff \text{ if } A \text{ is true then } B \text{ is true} \]

This gives the meaning of '→'. The 'if-then' on the right-hand side is the usual non-formal conditional. The meaning of 'A → B' is defined in the metalanguage of the formal language. This is unavoidable because, by Tarski's theorem on truth definitions, the truth predicate cannot be represented in a consistent formal theory. Therefore the meaning of 'A → B' must refer to something in the object language. But this contradicts the conclusion above that 'A → B' refers to something in S, that is, in the object world. When we reason directly about the system S, we only refer to S, its components and the properties, relations, and functions occurring in S. They all belong to the object world and not to the object language. Therefore the formal conditional 'A → B' cannot refer to anything in the object world. On the other hand, a semantic set-theoretical model of S contains all the truth-functions needed to define the meaning of 'A → B' as in (3-1). Similarly, the other connectives refer to functions at the object language level.

In the same way we see that the existential and universal quantifiers refer to functions at the object language level which can then only only belong to the model. Only entities at the object level in the model can represent something in reality. Then the quantifiers range over members of the domain of the model. To see this, we note that an existentially quantified sentence like \( \exists x \ A(x) \) just is a way to write a finitely or infinitely long disjunction with a disjunct for every element in the domain. If there are three elements a, b, c in the domain, then \( \exists x \ A(x) \) expresses

\[ A(a) \vee A(b) \vee A(c) \]
Since the existential quantifier is a generalised disjunction, it refers to a functional between entities ("truth-values") at the object language level. Since there are no such functionals at the object level and only the object level can be or represent reality, the functional must belong to a model. Then the existential quantifier ranges over the domain of the model. Similarly, the universal quantifier is a generalised conjunction which ranges over the domain of a model.

1.4 EXAMPLE (Atwood's Machine). In the gravitational field g of the Earth, two objects a and b, both of mass 4.00 kg, hang 1.00 m above the floor from the ends of a soft and inelastic cord, 4.00 m long and passing over a frictionless pulley. Initially, both a and b are at rest. We then have:

(4-1) If a moves downwards, then b moves upwards

The truth of this conditional is based on a function existing in the object world and determined by the experimental arrangement and especially the inelasticity of the cord. Generally, non-formal conditionals are based on functional connections in the object world. In contrast, the formal conditional

(4-2) a moves downwards $\rightarrow$ b moves upwards

is based on a truth-functional connection between 'a moves downwards' and 'b moves upwards' which only exists in a semantic set-theoretical model of the experimental arrangement but not in the Atwood machine itself. Therefore a sentence of the form (4-2) applies only to the set theoretical model while the sentence (4-1) applies to reality itself. Consequently, the non-formal logic for sentences like (4-1) applies directly to reality while the formal sentences like (4-2), and hence their formal logic, apply only to models.

1.5 EXAMPLE. Another example of how confused many logicians, mathematicians, and philosophers are about the question what formal logic can be applied to can be given. The following result for standard deduction systems for classical formal logic

(5-1) $A_1, \ldots, A_n \vdash B \Rightarrow A_1, \ldots, A_n \models B$

is called the Soundness Theorem. It is said to show that the deductive system can never lead from true premises to a false conclusion. If we inspect the proof, we see that it does not really show this but only the more restricted result that the deductive system can never lead from true premises to a false conclusion when applied to set-theoretical models. At the same time, most such logicians, mathematicians, and philosophers acknowledge the existence of systems which cannot be adequately represented in a classical set theoretical model and in which the classical Soundness Theorem (5-1) fails.

(I) The Law of Excluded Middle
(5-2) \[ \vdash A \lor \neg A \]
is classically provable. Intuitionistic Logic is the logic of constructive mathematics. The Law of Excluded Middle fails in intuitionistic logic. This shows that a classical deductive system can lead from premises which can be jointly true in intuitionistic mathematics to a conclusion which is false in intuitionistic mathematics. Therefore the Soundness Theorem of classical logic fails in constructive mathematics.

(II) The Distributive Law

(5-3) \[ A \land (B \lor C) \vdash (A \land B) \lor (A \land C) \]
is classically valid. But it fails in Quantum Logic based on the Hilbert space model of quantum mechanics. When the Distributive Law is combined with premises which are true in the Hilbert space model of quantum mechanics, it can lead to conclusions which are false in the same model. This shows that there are Hilbert space models of quantum mechanics which cannot be adequately represented in set-theoretical semantic models. Classical formal logic cannot be applied directly to such Hilbert space models and their corresponding quantum systems.

(III) The Law of Conjunction Introduction

(5-4) \[ A, B \vdash A \land B \]
is provable in classical logic, intuitionistic logic, and in the standard quantum logic. It is generally believed that this logical law cannot possibly fail. Nevertheless I show in my book *Logical Physics: Quantum Reality Theory* that if there is a local quantum reality, then the law (5-4) must fail there. Moreover, it can be shown that the failure of (5-4) is not as absurd as it first appears since the invalidity of (5-4) in the quantum world is a consequence of an ontic interpretation of the indeterminacy relations. If the logical law (5-4) is applied to premises which are true in a local quantum reality, this can lead to false conclusions like for instance Bell's inequality. Therefore classical formal logic is not sound when it is applied to a local quantum reality, and classical formal logic cannot be applied directly to a local quantum reality. It can only be applied to set-theoretical semantic models of such a kind of reality which can be adequately represented in such models.

1.6 ANALYSIS. We consider logics of truth, that is, such logics which apply to fields of sentences which can be assigned exactly one of the truth values True and False. Such a logic is formal iff every logically valid inference

(6-1) \[ A_1, \ldots, A_n \models B \]
is valid due only to the form of the sentences \( A_1, \ldots, A_n, B \) and not dependend on their contents. As examples of such inferences, we consider the Modus Ponens both in formal logic and in the nonformal logic used in ordinary language:

(6-2) \[ A, A \rightarrow B \models B \]
(6-3) \[ A, \text{if } A \text{ then } B \models B \]

The meaning of ‘\( \rightarrow \)’ is given by its truth-table and the validity of (6-2) can be shown by a truth-table. This proof also shows that (6-2) is valid due only to the form of the sentences A, A \( \rightarrow \) B, B. The 'if-then' of ordinary language has (almost) always the meaning:

(6-4) \[ \text{If } A \text{ then } B \iff \text{there is a function } f \text{ such that if } A \text{ describes an input to } f, \text{ then } B \text{ describes the corresponding output from } f. \]

The meaning assignment (6-4) validitates (6-3). But (6-3) is not formally valid as can be seen from (6-4). The validity of (6-3) depends on the content of A and B because A must describe an input to f and B must describe the corresponding output.

How is the step from nonformal logic to formal logic taken? In Hansen (1996b), I prove the following relation:

(6-5) \[ A \rightarrow B \text{ is true } \iff \text{if } A \text{ is true then } B \text{ is true} \]

The 'if-then' on the right hand side is the nonformal 'if-then' defined in (6-4). The truth-table for 'A \( \rightarrow \) B' together with the assumption that 'A \( \rightarrow \) B' is true determines a function which for input 'A is true' gives output 'B is true'. Equivalence (6-5) shows that there is a close relation between ‘\( \rightarrow \)’ and 'if-then'. Similar relations can be proven for the other connectives. This explains why formal logic is a good model of nonformal logic. In (6-5), we also note a difference. While ‘\( \rightarrow \)’ on the left hand side belongs to the object language, 'if-then' on the right hand side belongs to the metalanguage. This difference explains, as it turns out, that formal logic is not a perfect model of the nonformal logic of ordinary language. The **Equivalence Principle** (also called the **T-Principle**) is the sequence of statements

(6-6) \[ 'S' \text{ is true } \iff S \]

one for every sentence S with a definite truth-value. Applying the Equivalence Principle to (6-5), we get

(6-7) \[ A \rightarrow B \iff \text{if } A \text{ then } B \]

This is how the step from nonformal logic to formal logic is taken. The use of the Equivalence Principle leads to an identification of the nonformal conditional with the formal conditional. The equivalence (6-7) is obviously false because 'if A then B' implies the existence of a functional relation between the content of A and the content of B while 'A \( \rightarrow \) B' has no such implication. By pressing the relation (6-4) from the metalanguage down to the object language by the Equivalence Principle, we at the same time erase the functional relation between A being true and B being true which is present on the right hand side of (6-5). This is just what is needed to get a formal logic, that is, a logic where inferences like (6-1) are valid only due to the form of the sentences A\(_1\), \ldots, A\(_n\), B.
We consider a variant of an example invented by Cooper (1968). We start with the following simple and well-known equivalences from classical sentential logic:

(6-8) \( A \rightarrow B \Leftrightarrow \neg A \lor B \)

(6-9) \( A \lor B \Leftrightarrow B \lor A \) \( (\lor \text{ is commutative}) \)

(6-10) \( (A \lor B) \lor C \Leftrightarrow A \lor (B \lor C) \) \( (\lor \text{ is associative}) \)

From these preliminaries, we easily derive the equivalence

(6-11) \( (P \rightarrow F) \lor (R \rightarrow I) \Leftrightarrow (R \rightarrow F) \lor (P \rightarrow I) \)

and hence

(6-12) \( (P \rightarrow F) \lor (R \rightarrow I) \models (R \rightarrow F) \lor (P \rightarrow I) \)

Thus when \( \rightarrow \) is read as the formal truth-functional connective, \( [(R \rightarrow F) \lor (P \rightarrow I)] \) is a logical consequence of \( [(P \rightarrow F) \lor (R \rightarrow I)] \). Replace \( \rightarrow \) with 'if-then':

(6-13) \( \text{(if } P \text{ then } F) \lor (\text{if } R \text{ then } I) \models (\text{if } R \text{ then } F) \lor (\text{if } P \text{ then } I) \)

Interpret the sentential parameters in the following way:

\[
\begin{align*}
P &:= \text{I am in Paris} \\
F &:= \text{I am in France} \\
R &:= \text{I am in Rome} \\
I &:= \text{I am in Italy}
\end{align*}
\]

Then it is obvious that (if \( P \) then \( F \)) and (if \( R \) then \( I \)) both satisfy the truth condition (6-4) while neither (if \( R \) then \( F \)) nor (if \( P \) then \( I \)) do. Consequently, (6-13) is not valid. This shows that the nonformal logic of ordinary languages differs essentially from formal classical logic. Formalisation has a price. It is the identification, via the T-principle, of statement (6-7) in the object language with the statement (6-5) in the metalanguage that makes formal languages and formal logic possible. But since the meaning of the formal logical operators is given as referring to truth functions, that is, to something in the object language, and must be so given, and truth functions do not exist at the object level and therefore not in the relevant fragment of reality, formal languages and logic can only be understood as being concerned with models, and not directly with reality.

1.7 PROBLEM. The logical validity of (6-12) and the failure of (6-13) show that 'if-then' and '\( \rightarrow \)' do not have the same truth conditions. The left hand side and the right hand side of (6-5) provably have the same truth conditions. The two sides of the Equivalence Principle (6-6) also have the same truth conditions. Given this, how can the two sides of (6-7) fail to have the same truth conditions?

SOLUTION:
'A → B' in (6-7) and 'A → B is true' in (6-5) do have the same truth conditions. Similarly, 'A is true' and 'B is true' in (6-5) have the same truth conditions as A and B, respectively, in (6-7). But this does not suffice for the conclusion that then 'if A is true then B is true' and 'if A then B' have the same truth conditions. On the right hand side of (6-5), the functional connection between antecedent and consequence, demanded by Condition (6-4), is between A being true and B being true. By (6-4), this functional connection is an essential part of the truth condition for 'if A is true then B is true'. Applying the T-principle to 'A is true' and 'B is true', as is done in the step from (6-5) to (6-7), does not transport that functional connection down to 'if A then B'. Instead this operation erases the functional connection and reduces 'if A then B' to a formal conditional, just as is said in (6-7).

1.8 HISTORICAL REMARK (Formalisation). Three steps characterise jointly formal logic.

(8-1) The use of parameters and symbols to represent for instance sentences, predicates, functions, individuals, and logical operators. The purpose of this is to make the form of the sentences used in the logic appear clearer. Aristotle is credited for this invention. Some historians also claim that this shows that Aristotle invented formal logic. This is a mistake, however. The use of symbols does not suffice to make a language and a logic formal as should be evident from Analysis 1.6.

(8-2) The identification of object language and metalanguage via the Equivalence Principle, as in (6-5) and (6-7) above. Stoic logicians seem to have been the first to take this step which was later made more transparent by Frege, Ramsey, Tarski, Wittgenstein, Łukasiewicz, and Post. This step gives rise to formal logical principles.

(8-3) The integration of the formal logical principles into a calculus. This step was taken by Frege. Thus a long historical process is behind the development of the idea of formalisation; but I should say that the final step was taken by Frege.

1.9 NOTE. In the next three sections, I study some philosophical consequences of Observation 1.3.

2. Existence and Reference
2.1 Introduction. A considerable number of philosophers and logicians have wrestled with problems of existence and reference in formal logic. The discussions have often been centred around the existence of God and the reference to God by names and definite descriptions. They might as well have been centred around the existence of and reference to for instance centaurs or infinite sets. As examples of philosopher-logicians who treat such problems of reference and existence I take Russell (1910) and Quine (1951).

2.2 Constants. Constants are the formal analogues of proper names. First suppose we want to express statements about God's existence and non-existence, and we consider 'God' a proper name as it normally is construed. We introduce a constant 'g' into our formal language meant to refer to God. Then apparently we can express 'God exists' by

\[(2-1) \exists x \ x = g\]

and 'God does not exist' by

\[(2-2) \neg \exists x \ x = g\]

But (2-1) is a logical consequence of

\[g = g\]

which is a logical truth. Then also (2-1) is a logical truth, and the theistic standpoint cannot possibly be a logical truth. Similarly, the sentence (2-2) has as an immediate logical consequence

\[(2-3) \exists y \neg \exists x \ x = y\]

which can be read as

\[(2-4) \text{ There exists something which does not exist;}\]

or as

\[(2-5) \text{ There exists something which is different from everything (including from itself).}\]

As Quine rightly points out, this is "a contradiction in terms." Then the atheistic statement "God does not exist" is also a contradiction; but it must surely be possible to express the atheistic standpoint without contradiction.

2.3 Russell. I first briefly consider Russell's attempt to solve this problem. His first step is to consider 'God' a non-formal predicate instead of a proper name. In the formal language, he introduces a formal predicate 'G' with the intended interpretation

\[(3-1) \text{ G(x) } \leftrightarrow \text{ x is God, or} \]

\[\text{G(x) } \leftrightarrow \text{ x is godlike.}\]
He refers to God as an individual by a definite description 'that which is godlike' which formally can be represented by

\[(\exists x) G(x)\]

The theistic standpoint can be represented by

(3-2) That which is godlike exists

and the atheistic standpoint by

(3-3) That which is godlike does not exist

He then uses his method of contextual analysis of sentences involving definite descriptions. Thus the formal analysis of the sentence 'God exists' is

(3-4) \[\exists x (G(x) \land \forall y (G(y) \rightarrow y = x))\]

while 'God does not exist' is analysed as

(3-5) \[\neg \exists x (G(x) \land \forall y (G(y) \rightarrow y = x))\]

2.4 REMARK. (I) In ordinary language, 'God' is used as a proper name. The problem is to show how this procedure can be transferred to formal languages. Russell does not solve this problem. Instead he tries to circumvent it by using definite descriptions.

(II) The sentence (3-5) is logically equivalent with

\[\neg \exists x G(x) \lor \exists x \exists y (G(x) \land G(y) \land x \neq y)\]

which shows that the supposed atheistic statement (3-5) is compatible both with atheism but also with polytheism. This is not acceptable.

2.5 Quine. Quine follows Russell in considering 'God' a predicate and in referring to God as an individual by definite descriptions.

In predicate logic there are simple and straightforward ways to express the theistic doctrine that there is at least one god

(5-1) \[\exists x G(x)\]

the monotheistic doctrine that there is exactly one god

(5-2) \[\exists y \forall x (x = y \leftrightarrow G(x))\]

as well as the atheistic doctrine that there is no god

(5-3) \[\neg \exists x G(x)\]

We still want to be able to express that the individual which is God exists or, alternatively, does not exist. Like Russell, Quine's proposal for a solution is to represent the godlike individual by a definite description:
If there is a unique individual which is godlike, then

\[(5-4) \quad (\forall x) \ G(x) = \{\text{God}\}\]

If there is no godlike entity,

\[(5-5) \quad (\forall x) \ G(x) = \emptyset \]

where '∅' represents the empty set. If there is more than one godlike entity God₁, God₂, ..., Godₙ, then

\[(5-6) \quad (\forall x) \ G(x) = \{\text{God}_1, \text{God}_2, ..., \text{God}_n\}\]

Quine claims some advantages of his approach over Russell's. The definite description \((\forall x) \ G(x)\) always refers to one unique object irrespective of whether there is no god, there is one god, or there are several gods. We can use name of deities if only such names only occur under set brackets. The monotheistic standpoint can be represented by (5-2) or by (5-4), the atheistic standpoint can be expressed by (5-3) or by (5-5), and the polytheistic standpoint by (5-6).

2.6 REMARK. (I) Like Russell, Quine circumvents the problem by using definite descriptions rather than solving it. In ordinary language, 'God' is used as a proper name in the predicate logical language itself and not only under set brackets. Quine ought to show how this procedure can be transferred to formal languages. If this cannot be done, it would be an indication of a much larger difference between natural languages and formal languages than logic fans like Quine usually are willing to admit.

(II) From (5-4), (5-5) and (5-6), it is possible to derive by elementary logic the existence and uniqueness results

\[(6-1) \quad \exists y \ y = (\forall x) \ G(x)\]

\[(6-2) \quad \forall y \ \forall z \ (y = (\forall x) \ G(x) \land z = (\forall x) \ G(x) \rightarrow y = z)\]

as valid irrespective of whether there is no god, exactly one god, or several gods. From (5-4) and (5-5), we can see that what exists uniquely is either the set \{God\} or the empty set ∅ and similarly for (5-6). Suppose '(∀x) G(x)' follows the logic of definite descriptions in the classical sense. Then the individual denoted by '(∀x) G(x)' is determined uniquely as an entity satisfying the predicate 'G'. Therefore (6-1) implies

\[(6-3) \quad \exists y \ G(y)\]

which is incompatible with the claim that (6-1) is valid even when there is no god. Consequently, Quine's definitions (5-4) and (5-5) imply that Quine's '(∀x) G(x)' is not a definite description in the classical sense. The uniqueness clause (6-2) follows from (5-6) though uniqueness is incompatible with polytheism. The logic of definite descriptions excludes the possibility of making the stipulations (5-4) and (5-5). By that
logic, if there is a unique god, then \((\forall x) G(x)\) denotes that god. Otherwise, if there is no god or several gods, then the denotation of \((\forall x) G(x)\) cannot be defined.

(III) Quine's treatment of expressions like \('(\forall x) G(x)'\) involves the language of set theory and some set theory also. But a discussion of the existence or non-existence of God ought to be possible to conduct in a predicate logical language without any set theory at all.

2.7 SOLUTION. A solution is implied by Observation 1.3 to the effect that formal logic applies only to semantic set-theoretic models. Consider again the formal sentence

\[(7-1) \quad \exists x \ x = g\]

which is supposed to express the theistic standpoint. Since \((7-1)\) is a logical truth and the theistic standpoint is not, \((7-1)\) cannot express the theistic standpoint. By Observation 1.3, a formal sentence like \((7-1)\) refers to a model \(M = (M, g, \ldots)\) and not directly to reality. Then the constant \(g\) denotes an element \(g\) of \(M\), and then \((7-1)\) expresses the trivial truth that there is something in \(M\) which is identical with \(g\). Quantifiers range over the domain \(M\) of the model and not over reality. The existential quantifier \(\exists x\) expresses existence in the model, not existence in reality.

How do we express existence in reality in a formal language? By introducing a one-place existence predicate \(E\) with the intended interpretation

\[(7-2) \quad E(x) \leftrightarrow x \text{ represents an object which exists in reality}\]

The introduction of such an existence predicate is justified and meaningful when we operate in a model which is intended to represent (a fragment of) reality. Kant scared the philosophers into believing that existence is not a predicate. This may be justified when we talk directly about reality. But it is not justified when we talk about a model. Then it is meaningful and informative to have an existence predicate \(E\) with the interpretation \((7-2)\). Suppose a monotheist and an atheist discuss Gods existence. First they agree on the language \(L\) needed to refer to reality and on a model of (a suitable fragment of) reality:

\[M = (M, \ldots) = (\{a, b, \ldots\}, \ldots)\]

The dots stand for the properties, relations, individuals, and functions needed to model reality. The domain \(M\) contains only representatives of such individuals on the existence of which in reality the atheist and the monotheist agree. They also agree on which properties God must have if God there is. They use the constant \(g\) to denote this hypothetical being and the one-place predicate \(G\) to denote Gods divine properties. They extend the language to \(L' = L \cup \{g, G\}\). Next they extend the domain \(M\) to

\[M' = M \cup \{g\}\]
by adding a new element $g$ not occurring in $M$. Finally they expand the model $M$ by
adding interpretations $g$ and $G$ of the new symbols $g$ and $G$. Then we get a model

$$M' = (M', g, G, ...) = (M', g, \{g\}, ...)$$

While $M$ is supposed to be a model of reality, $M'$ has the character of a thought ex-
periment which the atheist and the monotheist agree to make. They have agreed to let $g$
represent God in the model, and they have agreed that if God exists, then God
should have the properties $G$. Finally, they extend the language once more by adding
the one-place predicate $E$

$$L'' = L' \cup \{E\}$$

with the intended interpretation (7-2), and they expand the model $M'$ further by add-
ing their interpretations of $E$. Now they do not agree any longer. The monotheist and
the atheist expand $M'$ in different ways by giving distinct interpretations to $E$:

(7-3) $M_{\text{atheist}} = (M', g, G, E, ...) = (\{g, a, b, \ldots\}, g, \{g\}, \{a, b, \ldots\}, ...)$

(7-4) $M_{\text{monotheist}} = (M', g, G, E, ...) = (\{g, a, b, \ldots\}, g, \{g\}, \{g, a, b, \ldots\}, ...)$

We have,

(7-5) $M_{\text{monotheist}} \models E(g)$

from which follows by logic

(7-6) $M_{\text{monotheist}} \models \exists x E(x)$

Though this sentence is true in $M_{\text{monotheist}}$, it is not logically true because it means:

(7-7) There exists something in the model which represents an entity that exists
in reality.

This follows because existence-in-the-model, represented by the existential quantifi-
er, is not the same as existence-in-reality, represented by the existence predicate. Similarly, the atheistic standpoint is true in the model $M_{\text{atheist}}$:

(7-8) $M_{\text{atheist}} \models \neg E(g)$

By logic,

(7-9) $M_{\text{atheist}} \models \exists x \neg E(x)$

In contrast to Quine's (2-3), this sentence is not a contradiction in terms because it
means

(7-10) There exists something in the model which does not represent an entity
that exists in reality.

Again the crucial point is the difference between existence-in-the-model, represented
by the existential quantifier, and existence-in-reality, represented by the existence predicate.
We can use constants to denote God without contradictions arising because formal logic is concerned with models and not directly with reality itself. A tiny bit of logic solves a problem which philosophers like Russell, Wittgenstein, Carnap, Quine, and several others found mind-boggling.

3. Ontological Commitment

3.1 Introduction. For many years ontology was banned from analytical philosophy. The common attitude was expressed by Gilbert Ryle in the slogan: "Ontologizing is out." The situation changed with Quine's idea of ontological commitment.

3.2 Ontological Commitment. Quine's ideas on ontology can be expressed in two principles. The first principle is the criterion of existence. This is expressed in a slogan:

(2-1) To exist is to be the value of a bound variable.

The thought here is that whenever we quantify over a kind of entities, then this kind of entities is explicitly or implicitly assumed to exist. In the simplest case, consider a sentence

(2-2) \( \exists x \ Q(x) \)

If the sentence is true, then this is claimed to imply the existence of an entity or a type of entities with the property Q. We note that it is the quantification, independent of whether it is existential or universal quantification, which by Quine is claimed to imply the existence of the elements in the domain quantified over. Thus the example ‘\( \forall x \ Q(x) \)’ should have worked as well. The second principle is the principle of ontological commitment. If a person P believes in a theory T and T involves quantification over a certain kind of entities, then P is, by his belief in T, committed to believe in the existence of this kind of entities.

3.3 EXAMPLES. (I) In Peano arithmetic PA, we quantify over natural numbers. This implies the existence of natural numbers in PA. If we believe in Peano arithmetic, then we are ontologically committed to believe also in the existence of the natural numbers.

(II) In the set theory ZF, there is a theorem which asserts the existence of sets of arbitrarily large infinite cardinality. Believing in ZF commits us ontologically to a belief in the existence of sets, and especially in the existence of sets of arbitrarily large infinite cardinality.
3.4 REMARK. If formal logic and formal theories had referred directly to reality, the idea of ontological commitment should have been correct. By Observation 1.3, they refer to models and not directly to reality. Therefore belief in a formal theory commits us to believe in the existence in any model of the theory of objects (sets) representing the kind of entities quantified over in the theory. Such a belief does not carry any commitment to belief in the existence in reality of the kind of entities quantified over in the theory. If we reconsider the false picture 1.1

```
Formal logic ———————— Reality
```

we see, from the falseness of the picture, that Quine was wrong when he placed ontological problems in the study of the relation between formal theories with formal logic on the one hand and reality on the other. In the correct picture 1.2

```
Formal logic ——— Set theoretical model ——— Reality
```

ontological problems do not belong to the relation between formal theories with formal logic on the one hand and set theoretical models on the other because existence in such a model does not carry ontological implications. If we want to ask ontological questions, we must either turn directly to reality or study the relation between semantic models and reality. This implies that the methodology of ontology is quite similar to the one used in classical ontology and has very little resemblance to the methodology proposed by Quine. From the analysis in Remark 2.7, we see that, given that we use formal logic and formal theories, the problems of existence and ontology arise in the relation between model and reality and neither in the relation between formal language and model nor in the direct relation between formal language and reality.

3.5 EXAMPLES. (I) Belief in Peano arithmetic PA commits us to belief in the existence of representatives of the natural numerals 0, 1, 2, … in every set theoretical model of PA. It does not commit us to a belief in the existence of natural numbers, neither in physical reality nor in a Platonic world. If we want to prove the existence of natural numbers, completely different methods must be used.

(II) Belief in the set theory ZF commits us to belief in the existence of representatives of all sets which can be proved to exist in ZF in any model of ZF. It does not commit us to a belief in the existence of sets, neither in physical reality nor in a Platonic world.

3.6 PROBLEM. Does not every consistent formal theory carry a limited ontological commitment? A set theoretical semantic model is built from sets. Therefore belief in a consistent formal theory seems to commit us to belief in the existence of sets. Some theories claim the existence of infinite objects. For instance ZF has an axiom which
says that there exists at least one infinite set. This infinite set must in any semantic model be represented by an infinite set so that we apparently become committed to a belief in the existence of infinite sets.

SOLUTION:

All hereditarily finite sets can be constructed or they can be assumed to exist. Therefore consider the question of the existence of hereditarily finite sets a non-problem. Infinite sets are in ZF the results of an existence claim (the Axiom of Infinity). In Hansen (2010), I show that it is possible to develop a set theory with infinite sets without any such existence claims. Instead an infinite set is the result of an assumption of the possibility (consistency) of a certain point of view. Assuming this means making a thought experiment. Since infinite sets come up only in thought experiments, they do not involve any existence claims. The resulting operational set theory is as powerful as ZF and is sufficient for model theory. For operational set theory, we are only committed to the existence of operations. By Turing’s thesis, operations can be identified with recursive functions. Operations exist in the physical reality.

3.7 CONCLUSION. Ontology cannot be performed the way suggested by Quine. The idea of ontological commitment is based on an elementary misconception of formal logic. Formal logic and formal theories do not apply directly to reality; they only refer to models. Therefore formal theories only carry a kind of existence commitments for models and have nothing to say about existence in reality. A tiny bit of logical understanding could have saved Quine from a philosophical blunder. There is no analytical ontology. Ontology has to be conducted by methods other than those of analytical philosophy. Since ontology is a fundamental part of philosophy, maybe even the fundamental part, analytical philosophy covers only a fragment of philosophy.

4. A Correspondence Theory of Truth

4.1 Introduction. The correspondence theory of truth says roughly:

(1-1) A sentence S is true if and only if S corresponds with reality.

There are other formulations preferring for instance that S is a belief, a proposition or a statement, and other formulations having that S should correspond with a state of affairs, a situation or a fact.

A main problem with all versions of the correspondence theory of truth for natural languages has been to define the correspondence relation. Though the correspondence theory of truth has a history of c. 2400 years, this problem still has no satisfactory solution. Tarski has, however, given a truth definition for formal predicate-logical
languages which defines precisely the correspondence relation. Below I try to extend Tarski’s truth definition to a correspondence theory of truth for a natural language like English.

4.2 DEFINITION (Tarski’s Truth Definition). I state what is essentially Tarski’s recursive truth definition for a given first-order predicate logical language with identity. I use the same notation as in my textbook Hansen (2003). Let \( L \) be the given language and let \( M = (M, ...) \) be a model for \( L \). Let \( L_M \) be \( L \) extended with a name of every element in \( M \). Expand \( M \) with interpretations of the names in the obvious way. I assume that the clauses for assigning values in \( M \) to predicates, constants, function symbols, and terms have already been defined. Then for all sentences (closed formulas) of \( L_M \), and hence for all sentences of \( L \):

\[
\begin{align*}
(2-1) \quad & M \models t_1 = t_2 \iff t_1 = t_2 \\
(2-2) \quad & M \models P(t_1, \ldots, t_n) \iff (t_1, \ldots, t_n) \in P \\
(2-3) \quad & M \models \neg B \iff M \not\models B \\
(2-4) \quad & M \models B \land C \iff M \models B \text{ and } M \models C \\
(2-5) \quad & M \models B \lor C \iff M \models B \text{ or } M \models C \\
(2-6) \quad & M \models B \to C \iff M \not\models B \text{ or } M \models C \\
(2-7) \quad & M \models B \iff C \iff M \models B \text{ and } M \models C, \text{ or } M \not\models B \text{ and } M \not\models C \\
(2-8) \quad & M \models \forall x \ B(x) \iff M \models B(a) \text{ for an arbitrary } a \in M \\
(2-9) \quad & M \models \exists x \ B(x) \iff M \models B(a) \text{ for at least one } a \in M
\end{align*}
\]

4.3 The Correspondence Theory. The truth definition 4.2 is a correspondence theory, and it gives a complete and precise definition of the correspondence relation. It has two drawbacks for our purposes:

\[
\begin{align*}
(3-1) \quad & \text{It is a theory of truth for formal languages only, not for natural languages.} \\
(3-2) \quad & \text{It is a theory of truth-in-a-model, not a theory of truth-in-reality.}
\end{align*}
\]

There is an obvious way to try to overcome these two difficulties. We use formal languages as models of natural, informal languages and formal logic as model of the non-formal logic of natural languages. There are well developed methods for translating between natural languages and formal languages. Similarly, we use set theoretic semantic models as models of reality. There are well developed methods for how to represent (fragments of) reality in a semantic model. We get the following picture:

Natural language —— Formal language —— Semantic model —— Reality

The formal language models the natural language. The semantic model represents (models) a fragment of reality. The truth definition 4.2 defines the correspondence
relation between the formal language and the semantic model. Observation 1.3 shows that the component *Semantic model* cannot be omitted from the chain because formal languages do not apply directly to *Reality*. This is an advantage for the theory of truth because semantic models, in contrast to reality, are so precisely defined that they make an exact definition of the correspondence relation possible. The presence of *Formal language* and *Semantic model* in the scheme will turn out to be not only an advantage but also the Achilles heel of the theory of truth. The use of models in solving the problem of defining truth is uncontroversial in itself. In physics, for instance, the use of models in problem solving is commonplace. Whenever a physicist wants to study a physical system, he first defines a mathematical model of the system. Then he studies the model and finally draws conclusions about the properties of the physical system from the properties of the mathematical model. The reliability of these conclusions depends, of course, on how well the model imitates the physical system.

This construction gives a correspondence theory of truth for natural languages. Those many who believe in formal logic should consider it an attractive and trustworthy correspondence theory of truth. The quality of the theory depends on how good formal languages and formal logic are as models of natural languages and their nonformal logic and on how good semantic models are as models of reality. I now give two examples which show that this correspondence theory is not sufficiently reliable to be a satisfactory theory of truth.

4.4 EXAMPLE. The following example is due to Cooper (1968). We use the sentential parameters

\[
\begin{align*}
P & := \text{I am in Paris} \\
F & := \text{I am in France} \\
I & := \text{I am in Istanbul} \\
T & := \text{I am in Turkey}
\end{align*}
\]

Consider the sentence (with brackets inserted for perspicuity):

(4-1) \quad [[\text{If I am in Paris then I am in France, and if I am in Istanbul then I am in Turkey}] \text{ implies that } [\text{either it is the case that if I am in Paris then I am in Turkey, or else it is the case that if I am in Istanbul then I am in France}]

Sentence (4-1) is clearly false. Now we use the truth theory 4.3. The first step is to formalise sentence (4-1) using the sentential parameters above. We get

(4-2) \quad [(P \rightarrow F) \land (I \rightarrow T)] \rightarrow [(P \rightarrow T) \lor (I \rightarrow F)]

which is a tautology. (Here we use the model relation between the informal sentence (4-1) and the formal sentence (4-2).) Then it is true in all semantic models. (Here we use the correspondence relation between formal sentences and semantic models.) By the theory of truth, we infer that the sentence is logically true (in reality). (Here we
use the model relation between semantic models and reality.) In other words, our correspondence theory of truth 4.3 implies that the false sentence (4-1) is logically true!

4.5 REMARK. (I) The example shows that 'if-then' and '→' do not have the same truth-conditions. The existence of one such example implies that there is an infinity of counterexamples which all show that the theory of truth 4.3 is invalid.

(II) The failure of the theory of truth 4.3 is a result of our having modelled the non-formal 'if-then' by the formal truth-functional '→'. The dominant reaction to this problem is to search for another formal logical system for the conditional → with a semantics and behaviour which better than the truth-functional conditional imitates the informal 'if-then'. Experience shows that no such system has been found in spite of much work and effort. It is possible to prove, as in Hansen (1996b):

\[(5-1) \quad A \rightarrow B \text{ is true } \iff \text{if } A \text{ is true then } B \text{ is true}\]

This shows that the truth-functional '→' is the closest we can get to 'if-then' in a formal logic. Therefore the theory of truth 4.3 cannot be saved by replacing classical formal logic by another formal conditional logic.

4.6 EXAMPLE. The following example is known as Ross's paradox and is due to Alf Ross (1941). From the imperative

\[(6-1) \quad \text{Post this letter!}\]

follows logically the imperative

\[(6-2) \quad \text{Post this letter or burn it!}\]

by applying the well-known tautology

\[(6-3) \quad \models A \rightarrow A \lor B\]

to the imperative sentences and assuming that '∨' adequately represents 'or'. The justification for the step from (6-1) to (6-2) is the principle:

\[(6-4) \quad \text{If a person P is ordered to see to it that a certain state of affairs } S \text{ becomes the case, then he is also implicitly ordered to see to it that all states of affairs which are logical consequences of } S \text{ become the case.}\]

The paradox arises as follows. The imperative (6-1) has the impact:

\[(6-5) \quad \text{Posting this letter is the only allowed alternative. Every alternative relevant action, including burning the letter, is excluded.}\]

The imperative (6-2) is by the argument above a consequence of (6-1). Therefore its impact is a consequence of (6-5):

\[(6-6) \quad \text{Posting this letter is an allowed alternative. Burning the letter is another allowed alternative. All other alternative relevant actions are excluded.}\]
The statements (6-5) and (6-6) contradict each other because burning the letter cannot both be allowed and not allowed. But it is a simple fact of logic that if a sentence is consistent, as (6-5) is, then every logical consequence of that sentence must be consistent with the sentence, which (6-6) is not.

The analysis makes it obvious that the paradox arises because we have assumed that \( \lor \) adequately represents 'or'. It does not. The impact of a non-formal disjunction

\[ A_1 \lor A_2 \lor \ldots \lor A_n \]

is that it gives an exhaustive and exclusive list of alternatives. Therefore we cannot add one more essentially different alternative

\[ A_1 \lor A_2 \lor \ldots \lor A_n \lor B \]

and expect to preserve truth. In contrast, the truth-functional disjunction

\[ A_1 \lor A_2 \lor \ldots \lor A_n \]

only expresses that at least one of \( A_1, A_2, \ldots, A_n \) is true. Therefore it is truth-preserving in this case to add one more sentence

\[ A_1 \lor A_2 \lor \ldots \lor A_n \lor B \]

Thus the erroneous step in the derivation of Ross's paradox is the identification of 'or' with \( \lor \) combined with the use of the formal logical law (6-3). The command (6-1) gives an exhaustive and exclusive list of the alternative actions allowed. Adding one more alternative, as in (6-2), therefore leads to contradiction with (6-1). The identification of 'or' with \( \lor \) is part of the correspondence theory of truth 4.3. This theory therefore has as a consequence that the statement (6-6) is true whenever (6-5) is true while, as a matter of fact, (6-6) is false whenever (6-5) is true.

4.7 CONCLUSION. The two examples, which can be multiplied indefinitely, show that the correspondence theory of truth 4.3 sometimes attributes the wrong truth-value to a sentence and hence to its associated proposition. Therefore it is not an acceptable theory of truth.

4.8 REMARK. As noted, it is possible to prove

(8-1) \[ A \rightarrow B \text{ is true} \iff \text{if } A \text{ is true then } B \text{ is true} \]

Similarly, it can be proven that

(8-2) \[ A \lor B \text{ is true} \iff \text{A is true or B is true} \]

This shows that the truth-functional \( \lor \) is the closest we can come to a faithful representation in a formal logic of the disjunction 'or' occurring in natural languages. A logic of commands and a deontic logic which avoids paradoxes of disjunction like Ross’s paradox must therefore be a non-formal logic.
5. What to Do Instead?

5.1 Logic. Observation 1.3 shows that a logic which can be applied directly to reality must be a non-formal logic of the sort used when we reason in a natural non-formal language. Logic has to do with thinking. When we think logically, we think correctly. When we think illogically, we think incorrectly. Thinking is problem solving in the head. There can be no thinking without problems. Therefore a theory of problems and problem solving is the right framework for the development of logic.

5.2 DEFINITION (Problem). A problem is a quest for something determined by a set of conditions. Therefore a problem can be written as

$$\Pi(C_1, \ldots, C_n)$$

where $C_1, \ldots, C_n$ are the defining conditions. A solution is anything which satisfies the conditions. Problems (quests) in this broad sense occur in all forms of life, from the lowest to the highest. In thinking, the quest is for information.

5.3 Problem Logic. I indicate some features of a logic developed within the framework of a problem theory.

(I) Conditionals. A non-formal conditional 'if A then B' is the solution to a problem $\Pi(C_1, \ldots, C_n)$ if the modified problem $\Pi(C_1, \ldots, C_n, A)$ has B as its solution. The analysis in Hansen (1996b) shows that the functional relations which take us from the conditions $C_1, \ldots, C_n, A$ to B must be functions at the object level, that is, at the level determined by the conditions $C_1, \ldots, C_n$, and not, as in formal logic, truth-functional relations at the object-language level, that is, one level above the level determined by $C_1, \ldots, C_n$. As a consequence, the implication paradoxes of formal logic are avoided.

(II) Disjunctions. A disjunction arises from alternative solutions to a problem. For instance, if $\Pi(C_1, \ldots, C_n)$ has A and B as its all and only solutions, then 'A or B' expresses just that, and 'A or B' is the complete solution to the problem. Thus 'A or B' gives an exhaustive and exclusive list of solutions to a problem which is understood in the context. This solves Ross's paradox and other paradoxes of disjunction in formal logic.

5.4 PROBLEM. If thinking is problem solving in the head, does not that exclude that the problem logic can be applied directly to reality? Must not all such supposed applications rather be to phenomenological models of reality in our minds?
SOLUTION:

We do think primarily in phenomenological models of reality; but via the Miss Rig-mor Holt projection function defined in Hansen (2007), these models are projected out on reality. Via the projection, our thinking is directed towards reality. When we perceive an external object as red, this shows that the object has a disposition to elicit the phenomenological quality of red in our minds. This disposition is a quality, red-ness, in the external object itself. Therefore language and logic can be concerned directly with reality and not only with phenomenological images of it.

5.5 Semantics. The considerations in §§ 5.1-5.3 presuppose a semantics. This semantics is indicated by the semantic square:

<table>
<thead>
<tr>
<th>Problem</th>
<th>Proposition (Proposed solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question</td>
<td>Sentence (Answer)</td>
</tr>
</tbody>
</table>

The Problem is a quest for information. The Proposition is a proposed solution to the problem. The problem can be expressed in a Question. The Sentence is a proposed answer to the question. The sentence expresses the Proposition.

5.6 Truth. From the semantic theory, we get a theory of truth. The defining conditions $C_1, \ldots, C_n$ of a problem $\Pi(C_1, \ldots, C_n)$ sometimes do not determine a unique solution. Therefore the complete solution to a problem is a set $\Sigma = \{P, Q, \ldots\}$ of propositions which must be read disjunctively: $S_P$ or $S_Q$ or $\ldots$. Here $P, Q, \ldots$ is an exhaustive and exclusive list of all solutions to $\Pi$, that is, $P, Q, \ldots$ and nothing else satisfy the conditions $C_1, \ldots, C_n$, and $S_P, S_Q, \ldots$ are sentences expressing the propositions $P, Q, \ldots$.

(6-1) The sentence $S$ is true in the context of problem $\Pi(C_1, \ldots, C_n)$ if and only if $\Sigma = \{P, Q, \ldots\}$ is the complete solution to the problem and $S = 'S_P$ or $S_Q$ or $\ldots'$ where $S_P, S_Q, \ldots$ are sentences expressing the propositions $P, Q, \ldots$. 
5.7 REMARK. The theory of truth 5.6 is, of course, in need of development. For instance the basis of all empirical knowledge is verified results of measurements of operationally defined entities. The projections onto reality of propositions representing such results are called facts. The sentences which express such propositions are the basic truths rather than the so-called “Protokollsätze” favoured by the logical empiricists. This is in agreement with the thesis that human beings are Turing machines and that therefore epistemology must be based on such operations which can be performed by the universal Turing machine.

5.8 Reference and Existence. A problem logic of the sort outlined above can be applied both to reality and to models. Therefore an atheist and a monotheist can conduct their discussion on the existence of God mainly as in the reconstruction 2.7. One difference is that now they must agree, explicitly or implicitly, that they are conducting a thought experiment. Another difference is that the models they use in the thought experiment are adapted to problem logic rather than to formal classical logic and therefore normally contain no truth-functions.

5.9 Ontological Commitment. Problem logic can be applied both to reality and to models. If we have a non-formal theory T with underlying problem logic and that theory by a person is claimed to apply to reality and be a true theory of reality, then this claim in itself implies a commitment of that person to the claim that whatever entity is implied by the theory to exist also exists in reality. If the non-formal theory T is applied to a model, the commitment is only to the existence of representatives of the named entities in the model. Ontological research cannot be done the way proposed by Quine. More in the vein of older philosophy, we must ask directly about the nature of reality. To move ontology from a speculative methodology towards a more analytical methodology, we can in the case of the ontology of physical entities take guidance of the fundamental theories of modern physics, quantum mechanics and a modified and corrected relativity theory. For the ontology of mathematical entities, some guidance comes from an analysis of the applications of mathematics.

5.10 Theory of Truth. In the present writer's opinion, the correct theory of truth for natural languages is the one given in § 5.6. It is not a correspondence theory of truth in the sense defined in Section 4, and it shows why a correct theory of truth for natural languages cannot be a correspondence theory. The reason is that truth is context dependent. A sentence gets a part of its meaning from the problem to which it expresses a proposed solution. It is true if it expresses a correct complete solution to that problem. Therefore the same sentence can be true in one problem context and false in another. In a correspondence theory of truth, any such context dependence is excluded.
5.11 **EXAMPLE.** Consider the problem

\[(11-1) \quad \Pi(x^2 - 3x + 2 = 0, x \in C)\]

It has the complete solution

\[(11-2) \quad x = 1 \text{ or } x = 2\]

Therefore the sentence (11-2) is true in the context of Problem (11-1). The sentence

\[(11-3) \quad x = 1 \text{ or } x = 2 \text{ or } x = 3\]

is not true in the problem context (11-1). This is because 'x = 1 or x = 2 or x = 3' is not a complete solution to problem (11-1) since 'x = 3' does not satisfy the first condition of the problem. Sentence (11-3) is false in context (11-1) because it fails as a complete solution to the problem (by not being an exclusive list). If we consider instead the problem

\[(11-4) \quad \Pi(x^3 - 6x^2 + 11x - 6, x \in C)\]

then the sentence (11-3) is true in the problem context (11-4) while sentence (11-2) is false (by not being exhaustive).

5.12 **REMARK.** Though the truth theory 5.6 is not a correspondence theory, the concept of truth given nevertheless contains the correspondence needed in some contexts. Suppose the problem \(\Pi(C_1, \ldots, C_n)\) is a quest for information about reality. Then some of the conditions \(C_1, \ldots, C_n\) must be concerned with reality. The correspondence goes from the sentence via the proposed solution and the conditions \(C_1, \ldots, C_n\) to reality. There is no direct correspondence between sentence and reality as there is in a correspondence theory of truth.

**NOTE.** In December 2009, I was invited to examine the second to last draft for Anders Kraal’s dissertation (2010) in Philosophy of Religion at Uppsala University. Observation 1.3 and the ideas on reference and existence in Section 2 occurred to me as a possible response to a chapter in the dissertation. I am indebted to Anders for a very stimulating discussion. The ideas and results on ontological commitment and correspondence theories of truth in sections 3 and 4 came later as corollaries to the observation. The extension in Section 4 of Tarski’s truth definition to a correspondence theory of truth for natural languages is original and may possibly be new. I first got the basic ideas on semantics, truth, and problem logic in Section 5 when I was sixteen and tried to solve the problem of semantic meaning. Much later I learned that R. G. Collingwood in the 1930s had developed partly similar ideas on meaning and truth.
References


Quine, W. V. O. (1951), Mathematical Logic. Harvard UP, Cambridge MA.


Ross, A. (1941), "Imperatives and Logic." Theoria, 7.