ABSTRACT. This essay is a critical analysis of some themes in Wittgenstein's later philosophy. It is not primarily Wittgenstein-exegesis. Much more modestly, my purpose is to express my own thoughts about some questions which Wittgenstein has treated in his writings. It is the second in a series of two articles. The first article, "Remarks on Wittgenstein's Philosophy: Private Language and Meaning", was published in Volume 42, 2007, of the present YEARBOOK.

Section 1, "Philosophical Method". Wittgenstein's conception of philosophy as language therapy is criticised. Instead philosophy is construed as foundational research. Wittgenstein's statement that mathematical logic cannot contribute to progress in philosophy is repudiated. Several examples of ideas and results in mathematical logic which have led to the solution of philosophical problems are given.

Section 2, "Contradictions: The Wittgenstein-Turing Debate". In lectures on the foundations of mathematics given in 1939, Wittgenstein claimed that contradictions in mathematical theories are harmless. A debate ensued on this question between him and Alan Turing. In support of Turing's standpoint, I use the theorem on Taylor series, Church's Theorem, and Gentzen's Cut-Elimination Theorem to show that Wittgenstein's standpoint is untenable.

For orientation, I also include here the abstract for the first article in the series, "Remarks on Wittgenstein's Philosophy: Private Language and Meaning".

Section 1, "The Private Language Argument". An independent argument is given for Wittgenstein's thesis that there is no private language. I show that psychological terms in ordinary language, in contrast to an implication of Wittgenstein's own private language argument, in their meanings do contain references to inner states, processes, and events.

Section 2, "Meaning". Wittgenstein's definition of meaning as use in the language is criticised. Meaning is instead identified with something in the content of a conscious mind. Applications are given to some suggestions in philosophy of language by Chomsky, Harman and Fodor, Grice, and Kripke.
1. Philosophical Method

1.1 INTRODUCTION. During my early undergraduate studies in philosophy, I was forced to learn something about Wittgenstein's philosophy, both from textbooks and by studying excerpts from the *Tractatus* and *Philosophische Untersuchungen*. I was quite dissatisfied with the ideas and opinions he expressed; but nothing irritated me more than his ideas on the therapeutic nature of philosophy, that philosophy really is language therapy.

(*In the present section, the quotations are my own translations into English from Wittgenstein's *Philosophische Untersuchungen.* *)

1.2 QUOTATION. "A philosophical problem has the form: 'I cannot find the way out.'" (§123)

"Philosophy must not in any way change the existing linguistic usage; it can after all only describe it. It cannot provide any foundations for it either. It leaves everything as it is." (§124)

"It is not the task of philosophy to dissolve a contradiction by a mathematical, a logical-mathematical discovery. The task is to make the state of mathematics — the state about which we feel troubled, the state *before* the dissolution of the contradiction — surveyable. (And this does not mean that we shun a problem.) The fundamental fact here is that we lay down rules, a technique, for a game and that then, when we follow the rules, things do not turn out as expected. That we are like caught in our own rules.

Our being caught in our own rules, that is what we want to understand, survey as it were." (§125)
"Philosophy simply places everything before us, and it explains or derives nothing. — When everything is transparent, there is nothing to explain either. Because what happens to be hidden does not interest us.

We might also call that 'philosophy' which is possible before all new discoveries and inventions." (§126)

"We do not want, in an unparalleled way, to refine or complete the system of rules for the use of our words. Because the clarity we are aiming at is in any case a complete clarity. But it only means that the philosophical problems should disappear completely.

The real discovery is the one which makes me capable of stopping the philosophising when I want to — the discovery which brings philosophy to rest so that it no longer is driven by questions which call itself into question. — Instead a method is displayed by examples, and the sequence of examples can be cut short. — Problems are solved (difficulties are eliminated), not a problem. There is not a method of philosophy but rather methods — like various therapies." (§133)

"The philosopher treats a problem — as were it a disease." (§255)

"What is your aim in philosophy? — To show the fly the way out of the fly-bottle." (§309)

1.3 REMARK. (I) Thus, in Wittgenstein's opinion, there are two kinds of philosophy. Traditional philosophy consists of ideas and theories based on misunderstandings of the proper way language works. The real philosophy is therapeutic philosophy. It consists in revealing and clarifying in what way the proper use of ordinary language has been violated in a theory or idea of traditional philosophy and in showing that if the pertinent words of ordinary language appearing in the theory are used correctly, then the traditional philosophical problem, to which the idea or theory was supposed to be a solution, disappears.
II) The following piece of motivation for the idea of therapeutic philosophy does not, as far as I know, occur in Wittgenstein's own writings. It nevertheless appears to me to be the reasoning which, wittingly or unwittingly, is behind Wittgenstein's view. There are some premises:

(3-1) All philosophical problems are purely conceptual problems.
(3-2) Concepts are meanings.
(3-3) Meaning is use in the language.

Premise (3-1) is a basic assumption in analytic philosophy. Wittgenstein has probably taken it over from Moore and Russell. Concerning (3-2), it is a common opinion among analytic philosophers that meanings and concepts are the same. Premise (3-3) is Wittgenstein's own theory of meaning. In Section 2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning", I showed that it is a consequence of the private language argument which is thus the support of the whole of Wittgenstein's later philosophy. When this argument does not hold, the whole edifice collapses.

Consider a philosophical problem \( \Pi \). By the first premise, \( \Pi \) is a purely conceptual problem. To solve it, we need only consider concepts. By the premises (3-2) and (3-3) together, concepts can be identified with use in the language. Therefore \( \Pi \) is a problem about the use in the language of the expressions in the formulation of \( \Pi \). The terms occurring in the formulation of \( \Pi \) are of two sorts: terms from ordinary language and technical terms. But to be intelligible, all technical terms must ultimately be definable in terms of ordinary language. Therefore the formulation of \( \Pi \) can be assumed to be in ordinary language. To clarify the conceptual issues in \( \Pi \), it is necessary and sufficient to clarify the proper use in ordinary language of the expressions occurring in the formulation of \( \Pi \) because this use exactly represents the meaning and hence the concept. The latter idea was effectively formulated by Wittgenstein's student Norman Malcolm in a slogan.
Proposition (Wittgenstein-Malcolm). Ordinary language makes no mistakes.

PROOF:

The impact of the statement is that no matter how an expression is used in ordinary language, this use cannot attach an incorrect meaning, and therefore not an incorrect concept either, to the expression. The reason is that the meaning is the use, the whole use, and nothing but the use in ordinary language.

Studying language use, we have now completely elucidated the concepts occurring in the formulation of the problem Π. Wittgenstein's contention is that with this clarification, the problem Π dissolves and disappears. It was a pseudo-problem and its dissolution results in no philosophical theory. To draw this conclusion, he apparently needs one more assumption:

(3-4) A body of knowledge about language-use is not a philosophical theory.

1.4 REMARK. (I) In my opinion, a philosopher should ask the fundamental questions about our existence. A philosopher should ask questions about reality, not only about language and language use. To philosophise is to strive for some understanding of this puzzling existence. This makes philosophy a great and worthwhile endeavour. Wittgenstein's view on philosophy, that its problems only are the result of linguistic misunderstandings, makes a trifling matter of it. To spend a life making philosophical problems evaporate by disentangling such misunderstandings is a sure way to waste one's life.

(II) In the essay "What is Philosophy?" in my book Applied Logic (1996), I investigate the nature of philosophy. The answer given is that philosophical problems are foundational problems; philosophy is foundational studies. The foundations can be of many different kinds: foundations of mathematics, of physics, of science in general; foundations of politics and of social organisation; founda-
tions of morals, of religion, of creative arts, and even foundations of life — to mention but a few. It is convenient to begin by studying the foundations of formalised theories of the sort considered in mathematical logic. The insights won by this study can then be generalised to other types of foundational studies.

Suppose we have a non-empirical problem $\Pi$. Sometimes such a problem can be solved in an established axiomatisable theory $T$. The solution consists in deriving in $T$ a solution $B$ to the problem. Sometimes the problem $\Pi$ cannot be solved in any existing theory. To solve the problem, we have to find one or more new, true assumptions $A_1, A_2, \ldots$ to add as axioms to an existing axiomatised and true theory $T$ such that $T \cup \{A_1, A_2, \ldots\}$ implies a solution to $\Pi$. If $\Pi$ is solvable in $T$, we have a recursive proof predicate by which the correctness of the solution can be checked. In the case where we need new assumptions $A_1, A_2, \ldots$, there is no recursive proof predicate by which $A_1, A_2, \ldots$ can be verified. We must use other methods, for instance conceptual analysis or speculation and insight. These are methods traditionally associated with philosophy. Since there are only two kinds of problems — those which can be solved in an existing theory and those which cannot — we can identify philosophical problems with the latter sort. But the basic principles of a theory are its foundations. Therefore philosophical problems are foundational problems. I give some examples of philosophical problems which are foundational problems and cannot be made to evaporate by analysis of the proper use of words.

(1) A theory $T$ which at first appears to be true turns out to be inconsistent. Find a way to revise the basic assumptions of the theory! An example is naive set theory. Cantor, Burali-Forti, and Russell derived antinomies in this theory. The solution by Zermelo was not found by analysing language, meaning, or concepts. It was found by studying the set formation process and gaining insight into the structure generated by this process, the cumulative type structure. This insight then became normative for our concepts of sets and set membership and
the use of the associated terms. The procedure was not the opposite, that the use in ordinary language of set terms determined the cumulative type structure, because ordinary language contains no clear rules for talking about infinite sets. Zermelo's solution is ontological rather than conceptual.

(2) A theory T can be incomplete in the sense that there is a problem Π which can be formulated in the language of T but to which no solution can be derived in T. Thus the continuum problem cannot be solved in ZFC (Zermelo-Fraenkel set theory with the axiom of choice). This shows that the continuum problem is a philosophical problem. For the same reason as in the preceding example, analysis of ordinary language cannot help here.

(3) Something about a theory is sometimes not well understood. Thus people have for centuries been puzzled by the amazing effectiveness of mathematics in physics. This is not a purely conceptual problem. It is an ontological problem which demands new insights into the nature of physical reality as well as a better understanding of mathematics.

(4) A theory sometimes does not satisfy a philosophical principle which is considered desirable. How to revise the theory? Thus Einstein's Special Theory of Relativity, in one interpretation, contains a convention, the so-called Einstein convention, which has observable consequences in the theory. A properly introduced convention should have no observable consequences. How to get an alternative theory of relativity without this undesirable feature?

(5) A theory may work very well in all applications. Nevertheless the theory is not understood. This is the case with quantum mechanics. Its predictions are verified with great precision; but it contains a host of predictions which are considered highly puzzling: indeterminacy, entanglement, apparent nonlocality, and the measurement problem, to mention but a few. How to develop the foundations of quantum mechanics to a point where these phenomena become intelligi-
ble and natural? Again this is an ontological problem and not a problem about the proper use of language.

(6) If ordinary language makes no mistakes and all questions of philosophy really are questions about language use, it is not possible to question the ordinary use of language. But this is often done in philosophy, for instance in applied ethics. In present-day established usage, abortion is not murder; but some people question this usage and see no ontological reason for drawing a dividing line between abortion and murder. Present-day established sexual morals accept sequential monogamy, that individuals have many lovers but only one at a time. Consequently, the term "infidelity" is not applied to sequential monogamy in present-day established usage. But it is possible to argue that this usage is based on arbitrary and conventional dividing lines which do not exist in reality. It reduces morals to rule following — like in tennis: when the ball is on one side of the line, it is *in*, and when it is on the other side, it is *out*. In other words, the established sexual morals and associated linguistic usage have no ontological foundations. This shows that ethical problems can be ontological rather than conceptual and linguistic.

(7) In Section 1 of my article "Remarks on Wittgenstein's Philosophy: Private Language and Meaning", I examined the private language argument. Wittgenstein was seemingly convinced that by examining the grammar of expressions in a language apparently concerned with states, processes, and events in the mind, he had showed that such expressions do not contain reference to (naming of) such states, processes, and events. We found that Wittgenstein's argument for his thesis is insufficient. We also found, by using facts about reality — for instance the role of the central nervous system in psychological states and processes and the existence of self-awareness — that psychological terms in their meanings do contain references to inner states, processes, and events in the mind.
We see that ontological questions are not conceptual problems and, in particular, are not questions about the correct use of words in ordinary language. Generally, ontology, epistemology, and ethics contain problems which cannot be solved by linguistic methods alone. The same is true of some of the problems in the foundations of the sciences and in philosophy of nature. *Ontological philosophy*, the conception of philosophical problems as being basically ontological, makes philosophy a worthwhile enterprise. By asking ontological questions, we ask the fundamental questions about reality and existence. Ontological philosophy avoids a frustrating feature of traditional philosophy, for instance analytic philosophy: the endless discussions which lead to no or almost no progress. The task of ontological philosophy is to build foundations. Foundations can be evaluated solely from the superstructure which can be erected on them. A working superstructure validates the foundations. New foundations of mathematics should lead to new (and possibly better) mathematics; new foundations of music should lead to new music, etc., etc.

1.5 ANALYSIS. If Wittgenstein's idea about philosophy as language therapy is wrong, then one or more of the premises (3-1) through (3-4) must be false. In the present context, I will accept the premise (3-2) to the effect that concepts are meanings. I look at the other premises.

*Premise (3-4):* This premise is false. Counter-instances can be found in Wittgenstein's own later work. He claimed that he did not put forward any philosophical theories or theses but only left reminders (about language use). This is not true. He did not himself live according to his own teaching. Thus, for instance, the private language argument leads to his claiming and defending two philosophical theses, labelled (4-1) and (10-1) in the exposition in Section 1 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning". He proposed a theory of meaning. He claimed that language therapy is the right way to do philosophy. (This is metaphilosophy; but metaphilosophy is also philosophy.) They
are also philosophical theses. In contrast, philosophy as foundational research admits and demands theses and theories.

Premise (3-3): The assumption of meaning as use in the language leads to Malcolm's contention that use always represents a correct conceptualisation. We saw in Section 2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning" that meaning cannot be identified with use. This opens for the possibility that we can question whether a given usage really represents the underlying concept adequately. This kind of calling in question is naturally considered to be philosophical. Example (6) in § 1.4 gives two cases in point from applied ethics. The general pattern is the following. Trying to understand the world, we try to identify sets of entities which are, by natural boundaries existing in the world, separated from other entities. Insights about such natural boundaries are philosophical (actually ontological) knowledge. Such insights may go beyond or question the correctness of previous insights. With the new insights, the perception of the world changes. Then also the prelinguistic concept and meaning change and the old use of the corresponding term is no longer considered adequate and is open to criticism.

Premise (3-1): This is the most difficult premise to discuss. The reason is the vagueness in the predicate "conceptual". Nevertheless I will claim that at least some ontological problems are not purely conceptual. According to Occam's Razor, one should not assume the existence of more types of ontological entities than are needed. A case of the use of Occam's Razor is Einstein's Special Theory of Relativity. In Fitzgerald's, Larmor's, and Lorentz's theories of electrodynamics, an ether is postulated. Einstein realised that this ontological entity could be eliminated and at least as good a theory of electrodynamics obtained by introducing instead the principle of the invariance of the speed of light and the idea of relative motion. But Occam's Razor, being a normative principle, is not purely conceptual. In my book from 1996, Logical Physics: Quantum Reality Theo-
ry, I investigate the ontological foundations of quantum mechanics. I find that in order to get a realistic and local quantum mechanics, the underlying ontology must be operationally defined. Here I use locality and realism as normative principles to select an operational ontology instead of, for instance, an ontology based on space and time. Again the demands of locality and realism being normative cannot be purely conceptual. The question of the correctness of Church's thesis is another philosophical problem which can be adduced as a counterexample to Premise (3-1). If the thesis is false, this can be shown by a purely conceptual argument. Just find a function which, on the basis of the concept of 'computable function', can be seen to be computable but which is provably not recursive. But if Church's thesis is true, as it appears to be, then this cannot be shown on purely conceptual grounds because the thesis equates the extensions of an empirical predicate ("computable function") and a mathematical predicate ("recursive function").

1.6 QUOTATION. "Philosophy must not in any way change the existing linguistic usage; it can after all only describe it. It cannot provide any foundations for it either. It leaves everything as it is.

It also leaves mathematics as it is, and no mathematical discovery can make it progress. A 'main problem in mathematical logic' is for us a mathematical problem like any other." (§124)

1.7 REMARK. A consequence of the last two sentences is that no discovery in mathematical logic can make philosophy progress. I now give some examples where results and ideas in mathematical logic have carried philosophy (in my sense) forward.

1.8 THEOREM (Soundness and completeness). Let T denote a theory and B a sentence in the language L(T) of T.
(8-1) \( T \) is consistent \( \iff \) \( T \) has a model.
(8-2) \( B \) is a theorem of \( T \) \( \iff \) \( B \) is true in all models of \( T \).
(8-3) \( B \) is provable in pure logic \( \iff \) \( B \) is true in all models of \( L(T) \).

1.9 REMARK. (I) Theorem 1.8-1 is useful in many contexts in philosophy. In philosophy of science, for instance, it gives a simple way of showing that a given theory is consistent: just find a model of the theory.

(II) Theorem 1.8-3 is extremely useful in investigations of the foundations of logic. It says that \( B \) is provable in the usual deductive systems for classical logic precisely in case \( B \) is logically true. This shows that the foundations of the deductive systems are sound and complete relative to the intended interpretation. It is nonsense to say that this is not an interesting and useful piece of information in the philosophy of logic. It has been questioned whether classical logic is sound in all applications, for instance in constructive mathematics and in quantum mechanics. A consequence of Theorem 1.8-3 is that this is only possible if constructive mathematics and quantum mechanics contain structures which cannot be adequately represented in the kind of set theoretical models used in classical logic. Again this is a useful piece of information in the philosophy of constructive logic and of quantum logic.

1.10 THEOREM. Let \( T \) be an axiomatisable theory containing Peano arithmetic.

(I) (Gödel's first incompleteness theorem). There is a sentence \( G \) such that:

(10-1) If \( T \) is consistent, then \( T \mid \not \models G \).
(10-2) If \( T \) is \( \omega \)-consistent, then \( T \mid \not \models \neg G \).

(* \( G \) expresses, interpreted in the metalanguage, "I, G, am not a theorem of \( T \).")
(II) **(Gödel's second incompleteness theorem).** Let \( \text{Thm}_T(x) \) express that \( x \) is the Gödel number of a theorem of \( T \). Let \( \text{Consis}_T \) be \( \neg \text{Thm}_T(\|0=1\|) \). Then:

\[
\text{(10-3) If } T \text{ is consistent, then } T \nvdash \text{Consis}_T.
\]

(* \( \text{Consis}_T \) expresses, interpreted in the metalanguage, that \( T \) is consistent. If \( S \) is an expression, \( |S| \) denotes the Gödel number of \( S \).*

1.11 **Hilbert's Program.** (I) Gödel's theorems were replies to Hilbert's program. Hilbert's purpose was to develop secure foundations for classical mathematics. Let \( T \) be an axiomatisable theory of classical mathematics which contains PA (Peano arithmetic). Hilbert divides \( T \) in a **finitistic** part and an **abstract** part. The abstract part contains the sentences and proofs in \( T \) which can only be defined by explicit or implicit reference to the actual infinite. The finitistic part contains the sentences and proofs in \( T \) which can be defined without reference to the actual infinite. (This is Hilbert's *semantic criterion* of finitism.) The finitistic part only needs the potential infinite as scope for its quantifiers, as domains and co-domains for its functions, and for induction proofs. The finitistic part of mathematics is the important part, the **real mathematics**, since it can be argued that applied mathematics is finitistic. The abstract part of classical mathematics is not strictly needed, Hilbert believed; but it eases the development of theorems of finitistic mathematics and makes the development more smooth running. A finitistic sentence was said by Hilbert to have the form \( \forall x_1 \ldots \forall x_n \ C(x_1, \ldots, x_n) \), which we abbreviate as \( \forall x \ C(x) \), where \( n \geq 0 \) and \( C(x) \) represents a decidable relation. (This is Hilbert's *formal criterion* of a finitistic sentence and believed by him to be equivalent with the semantic criterion.) Universal quantifiers must be allowed because a formula \( f(x) = G(x) \) in a mathematical handbook must be valid at least for all finitistic values of \( x \), \( \forall x \ (f(x) = G(x)) \), for instance for all rational numbers. (Hilbert believed, erroneously as it turned out, that all such sentences \( \forall x \ C(x) \) are finitistically provable.) On the other hand, \( \exists x \ C(x) \) is not
finitistic in itself. If $\exists x \ C(x)$ can only be proven by an indirect proof, then there is no guarantee that the object which satisfies $C(x)$ is finitistic. But if there is a direct, finitistic proof of $\exists x \ C(x)$, then it is inferred from $T \vdash C(e)$ for some finitistic $e$. In that case, $C(e)$ can be considered the finitistic form of $\exists x \ C(x)$, and $C(e)$ is of the form $\forall x \ C(x)$ with $n=0$.

What Hilbert believed and wanted to prove was:

(11-1) If $B$ is a finitistic theorem of $T$, then $B$ is true.

(11-2) The finitistic part of $T$ is self-contained: Every finitistic theorem of $T$ (in the formal sense) has a finitistic proof in $T$.

Hilbert proposed that logicians and mathematicians should first try to prove the following for pertinent theories $T$ of classical mathematics — such as Peano arithmetic, real analysis, complex analysis, and set theory:

(11-3) $T$ is complete.

(11-4) $T$ is consistent, by finitistic methods alone.

The central parts of Hilbert's program are the distinction between real and abstract mathematics and the purpose of proving the conservativity result (11-2); but Hilbert clearly found both of the goals (11-3) and (11-4) desirable, and (11-4) moreover necessary. One way to prove the truth result (11-1) and the conservativity result (11-2) is to derive them from (11-3) and (11-4). Hilbert's reason to prefer (11-3) and (11-4) over (11-1) and (11-2) as problems to attack first was that (11-3) and (11-4) looked more tractable. It can be argued that all valid finitistic proof methods are present in Peano arithmetic and therefore are available in $T$. I now show that (11-1) and (11-2) follow from (11-3) and (11-4). To derive (11-1), suppose that $B$ is a finitistic theorem of $T$ which is not true. As explained above, $B$ is of the form $\forall x \ C(x)$ where $C(x)$ represents a decidable relation. Thus the hypothesis is that
and ∀x C(x) is false. Then there is a finitistic e in the range of ∀x which makes C(x) false so that ¬C(e) is true. Since C is decidable, ¬C is. The calculation which shows ¬C(e) is a finitistic procedure. Since T is complete and contains all finitistic proof methods,

(11-6) \( T \vdash \neg C(e) \)

But from (11-5), we get

\[ T \vdash C(e) \]

which together with (11-6) implies that T is inconsistent, contrary to Assumption (11-4). Therefore \( B = \forall x \ C(x) \) is true if \( T \vdash B \) which proves (11-1). Define in \( T \):

\[ P(x, y) \iff x \text{ represents in } T \text{ a proof of a sentence represented by } y \]
\[ \text{Thm}_T(y) \iff \exists x \ P(x, y) \iff y \text{ represents a theorem of } T \]

Then we have just proved from (11-3) and (11-4) by a finitistic proof:

\[ \text{Thm}_T(|B|) \rightarrow B \text{ is true, if } B \text{ is finitistic} \]

Since T is consistent and complete and contains all finitistic proof methods,

(11-7) \( T \vdash \text{Thm}_T(|B|) \rightarrow B \) (* provided B is a finitistic sentence *)

by a finitistic proof in T. To see this, suppose (11-7) is false. Then by completeness,

\[ T \vdash \text{Thm}_T(|B|) \text{ and } T \vdash \neg B, \text{ for some finitistic } B \]

By (11-1), \( \text{Thm}_T(|B|) \) is true, that is, \( T \vdash B \) which is incompatible with the consistency of T. To see that (11-7) has a finitistic proof in T, we note that \( \text{Thm}_T(|B|) \) is decidable because T is consistent and complete. If \( \text{Thm}_T(|B|) \) is false, then the calculation gives a finitistic proof of \( T \vdash \neg \text{Thm}_T(|B|) \) from which (11-7) follows by sentential logic. If \( \text{Thm}_T(|B|) \) is true, the calculation
gives a finitistic way of proving $B$ in $T$, $T \vdash B$, from which (11-7) again follows by sentential logic.

To derive (11-2) from (11-3) and (11-4), let $B$ be a finitistic sentence which is a theorem of $T$. Then there is a derivation of $B$ in $T$. Any derivation in $T$, whether abstract or finitistic, is a finite object and therefore represented by a finitistic term $d$ in $T$. Hence $P(d, |B|)$ is true. Since $P$ is decidable, there is a finitistic proof of $P(d, |B|)$ in $T$:

$$T \vdash P(d, |B|)$$

Then, by a direct proof,

$$T \vdash \exists x P(x, |B|)$$

and hence by a finitistic proof

$$T \vdash \text{Thm}_T(|B|)$$

Then, by (11-7), we have by a finitistic proof

$$T \vdash B$$

which verifies (11-2).

This then is Hilbert's program: Prove (11-3) and (11-4)! — These results will verify (11-1) and (11-2) which in turn completely justify the use of abstract methods in classical mathematics. This follows since (11-1) and (11-2) show that such abstract methods can never result in any false theorems of real mathematics, and they show that though abstract methods may be convenient and time saving, we can always do without them if required. Hilbert starts with a certain philosophical conception of mathematics. Given this conception, he formulates the problem of the foundations of mathematics as a problem of mathematical logic.

Gödel's incompleteness theorems show that Hilbert's program cannot be realised. Let $T$ be an axiomatised and consistent theory of classical mathematics.
containing Peano arithmetic. Gödel's first theorem shows that T is not complete so that (11-3) is false and unprovable. Moreover G is, according to Hilbert's formal criterion, a finitistic sentence so that T is not even complete as far as true finitistic sentences are concerned. The second theorem shows that T cannot be proven consistent using only the methods available in T. Since T contains all finitistic proof methods, T cannot be proven consistent by finitistic methods alone. Even Consis_{T} is a finitistic sentence, as defined by Hilbert's formal criterion. A consequence of Gödel's theorems is that Hilbert's philosophical conception of mathematics is untenable and must be revised or given up all together. This is a philosophical consequence. The dominant opinion is that Gödel's theorems completely ruin not only Hilbert's program but also Hilbert's conception of mathematics. I do not agree with this. In my opinion, there are some sound components in Hilbert's philosophical conception of mathematics which should not be forgotten. Therefore a revision of Hilbert's conception is more relevant. In a revision process, Gödel's incompleteness theorems and recursion theory will be invaluable tools.

(II) Gödel's first theorem shows that mathematical truth (arithmetical truth, the truth of mathematical analysis, set theoretical truth) cannot be completely axiomatised. This is an epistemological result. Similarly, Gödel's second theorem to the effect that T cannot prove its own consistency is an epistemological result. The sentence G in the first theorem is not a theorem of T as shown in (10-1). But this is precisely what G expresses. Therefore G is true and ¬G false. This shows that mathematical sentences can have truth-values, in contrast to a widespread philosophical opinion. G and Consis_{T} are self-referential. The self-reference is constructed by the Gödel numbering. Self-awareness and self-consciousness are cases of self-reference. Gödel numbering gives a path to a better understanding of self-awareness and self-consciousness which are important in philosophical anthropology (for instance as used in Section 1 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning"). Self-reference may also be useful
in epistemology. In the essay "Logical Rationalism: A Program" in my book *Applied Logic* from 1996, I have suggested that *apriori truth* should be defined as truth in all self-referential models. This might give a way to prove the existence of apriori truth. Details can be found in the essay.

(III) In the first theorem, neither G nor \( \neg G \) is a theorem of T. Then the problem whether G is valid or not is a philosophical problem. A new insight beyond the information in the axioms of T is needed. The interpretation of G as "I, G, am not a theorem of T" together with the remarks in Point (II) above show that this insight can be reached by self-consciousness. Thus self-consciousness can be a source of non-analytic philosophical insight. The inexhaustibility theorem shows that the foundations can never be complete. Since philosophy is foundational studies, this implies that there is not and never can be any "end of philosophy". Moreover, since G is not a theorem of T, G is not an analytical consequence of the axioms of T. Nevertheless it is possible by philosophical insight to see that G is true. This shows that conceptual analysis is incomplete as a method of philosophy. There are solvable problems of philosophy which cannot be solved by analysis alone. Therefore analytical philosophy is inadequate as an ideology of philosophy. Other methods than philosophical analysis must be allowed in philosophical research.

(IV) Wittgenstein has, in *Remarks on the Foundations of Mathematics*, a discussion of Gödel's theorems. A main point is his calling in question whether it is meaningful to say that a sentence like G is true without a proof of G. (This criticism is concerned with the interpretation of Gödel's theorems and does not touch upon the mathematical correctness of the theorems.) The answer is that it is meaningful for the following two reasons.

(1) G is of the form \( \forall x \ C(x) \) where C(x) is a formula representing a recursive set. The algorithm for C(x) can be used to prove C(n) for each natural number n. Then C(n) is true for any n and provable in PA for any n, PA \( \vdash C(n) \), and
therefore \( \forall x \, C(x) \) is true because \( \forall x \) is meant to range over all the natural numbers and only over the natural numbers. This insight can be generalised and expressed in the infinitary so-called \( \omega \)-rule for an arithmetical theory \( T \):

\[
\text{From } T \vdash A(0), \, T \vdash A(1), \, T \vdash A(2), \ldots \text{ infer } T \vdash \forall x \, A(x)
\]

The \( \omega \)-rule resembles the induction principle. The \( \omega \)-rule is, however, much more powerful than the induction principle. Arguably, the main difference between them is that the induction principle is based on the potential infinity of the sequence of natural numbers while the \( \omega \)-rule is based on the actual infinity of this sequence. It is easy to prove by induction on the length of formulas that \( \omega \text{-arithmetic} = [\text{PA} + \text{the } \omega \text{-rule}] \) is complete arithmetic. Then all arithmetical sentences have a truth-value, in contrast to a wide-spread philosophical opinion. This is still another example of a philosophical consequence of a logical result.

(2) There is also another way to show that \( G \) is true. The way used in Point (II) above to show \( G \) true was to point out that \( G \), interpreted in the metalanguage, expresses about itself that it is not provable in \( T \). Since this is precisely what is proven in Gödel's first theorem, \( G \) is true (if \( T \) is consistent). This is possible because the Gödel numbering makes the formal theory \( T \) a model of \( T \) and in that model \( G \) is true. Let \( M \) be the class of all set-theoretical models of \( T \). Let \( J \) be the class of all set-theoretical models of \( T \) in which \( G \) is true, and let \( K \) be the class of all set-theoretical models of \( T \) in which \( \neg G \) is true. Then \( M = J \cup K \).

Then all models in \( K \) can be deleted from \( M \) by adding \( G \) as an axiom to \( T \), and all models in \( J \) can be deleted from \( M \) by adding \( \neg G \) as an axiom to \( T \). But \( T \) itself is a real and concrete model of \( T \) and cannot be deleted in the same way as the fictive and abstract models in \( M = J \cup K \). Since \( G \) is true in \( T \) (considered as a model of \( T \)), \( G \) must be considered to be really true and \( \neg G \) really false.

1.12 THESES. (I) (Church's thesis). Let \( f: \mathbb{N}^n \to \mathbb{N} \). Then
\( f \) is computable \( \iff \) \( f \) is recursive

(II) (Turing's thesis). Every algorithm, whether numerical or non-numerical, can be represented by a recursive function.

1.13 THEOREM. Let \( f: \mathbb{N}^n \to \mathbb{N} \). Then

\[ f \text{ is recursive } \iff f \text{ is Turing computable} \]

(* \( f \) is Turing computable \( \iff \) \( f \) is computable by some Turing machine.*)

1.14 REMARK. (I) The question concerning what information about a function can be attained by computation is an epistemological problem. By Church's thesis, such problems can be solved mathematically in recursion theory. By Turing's thesis, the same is true of epistemological questions about arbitrary algorithms.

(II) Church's thesis is not a mathematical result. It can be considered a philosophical thesis. If we combine Church's thesis and Theorem 1.13, we get the following equivalent form of Church's thesis:

(14-1) \( f \) is computable \( \iff \) \( f \) is Turing computable

While it is hard to give arguments directly for the original form of Church's thesis, Turing has given quite persuasive analytic arguments for the form (14-1). Thus Theorem 1.13 has transformed Church's thesis to a form which is more susceptible to philosophical arguments.

(III) Turing machines are useful models of living organisms and subsystems of organisms. In particular, they are useful models of the human brain. Turing machines are also models of computers and therefore part of the foundations of computer science.

(IV) In the traditional theories of definitions, a real definition of an entity states the essential properties of the entity. In contrast, a nominal definition states the
meaning of a word. According to modern epistemology and theory of definitions, there are no real definitions. However, Church's thesis can be adduced as an example of a real definition. Combined with the standard definition of recursive functions, it states the essential mathematical properties of computable functions.

1.15 THEOREM (Church). Let \( L = \{0, S, +, \cdot, <\} \).

(I) The set of theorems of Peano arithmetic PA in L is not recursive.

(II) The set of theorems of the predicate calculus for L is not recursive.

1.16 REMARK. (I) Church's theorem expresses that theoremhood in PA and in the predicate calculus for L cannot be decided by recursive methods. If we combine Church's theorem with the Church-Turing thesis, we get:

(16-1) Peano arithmetic is undecidable.

(16-2) The predicate calculus for L is undecidable.

These sentences express that theoremhood in PA and in the predicate calculus for L cannot be decided in general by any method. All these results are epistemological results. They are proved in mathematical logic, and they can only be proven by methods of mathematical logic.

(II) It can be argued that epistemological problems concerned with what is in principle knowable or unknowable can only be solved by recursion theory. Epistemology is a branch of applied recursion theory. The idea here is that a human being essentially is a complex Turing machine. All epistemic processes in a person are processes in a Turing machine. Then the theory of Turing machines and recursive functions is the right framework for the study of epistemic processes and their results, that is, epistemology. Some questions concerned with what
types of problems are in fact tractable or untractable can be solved in complexity theory and in chaos theory.

1.17 THEOREM. Let ZF be Zermelo-Fraenkel set theory and ZFC be ZF with the axiom of choice. Let CH designate the continuum hypothesis.

(I) (Gödel). If ZF is consistent, then CH is consistent with ZFC.

(II) (Cohen). If ZF is consistent, then ¬CH is consistent with ZFC.

1.18 REMARK. Jointly, the two theorems imply that the continuum problem cannot be solved in the established set theory ZFC. This implies that the continuum problem is a philosophical problem rather than a mathematical problem. The problem is to obtain such new insights into the nature of sets and the set universe (the cumulative type structure) which will allow a solution. One possible kind of solution consists in insights which together with ZFC imply the assignment of a definite truth-value to CH. Another possible solution might consist in showing that the situation in set theory is similar to the one in geometry which is said to allow both Euclidean and non-Euclidean geometries. A third possibility might be that though the Law of Excluded Middle implies that CH has a definite truth-value in the cumulative type structure, the definition, and hence the concept, of this structure nevertheless does not contain the information needed to determine the truth-value of CH.

1.19 REMARK. Gödel's theorem and Church's theorem give particularly good illustrations of the interplay between philosophy and mathematical logic.

(I) Hilbert wanted to justify the use of the abstract methods of classical mathematics. This is an epistemological problem. It is a problem outside every established theory and therefore outside the scope of every established recursive proof relation. To develop foundations for a recursive proof relation for the
problem, he developed a philosophical picture of classical mathematics. This leads to the formulation of the justification problem as two problems in mathematical logic: (1) Prove by finitistic (that is, indubitable) methods that PA and other theories of classical mathematics are complete, and (2) prove by finitistic methods that PA and other theories of classical mathematics are consistent. By philosophical reflection, Hilbert had brought a philosophical problem within the range of a mathematical theory with a well-defined proof relation, namely mathematical logic. Eventually, Gödel proved that the two problems of mathematical logic formulated by Hilbert cannot be solved. This has philosophical consequences back on Hilbert's thought: (1) Classical mathematics cannot be justified the way Hilbert hoped. (2) Hilbert's philosophical picture of mathematics is not tenable and must be abandoned or modified.

(II) Church's theorem starts with two problems of mathematics and of epistemology: (1) Is Peano arithmetic PA decidable? (2) Is the pure predicate calculus for the language L(PA) of Peano arithmetic decidable? Again we have problems without well-defined foundations and therefore without a well-defined recursive proof relation on which a definite solution can be based. The philosophical part of the problem consists in establishing such foundations. This part is solved by a philosophical thesis, Church's thesis. Using this philosophical thesis, the original problems can be reformulated: (1*) Is the set of theorems of PA recursive? (2*) Is the set of theorems of the pure predicate calculus for the language L(PA) recursive? This move brings the problems within the range of mathematical logic and a well-defined recursive proof relation. Church's theorem solves problems (1*) and (2*) definitely. The original problems (1) and (2) are solved to the extent in which Church's thesis is true.

(III) Philosophical problems are problems outside all established and well-defined foundations and therefore outside all established and well-defined recursive proof relations. Only problems which are solvable on the basis of an estab-
lished recursive proof relation (or, more generally, an established recursive ver-
ification relation) are scientific. This makes the idea of a scientific philosophy —
eagerly advocated by Carnap, Quine, and many other philosophers — an illu-
sion. The foundations of science cannot itself be a science. 'Scientific philos-
ophy' is a contradiction in terms. A scientific problem presupposes an established
recursive proof relation, or an established recursive verification relation, on the
basis of which it can be solved. A philosophical problem presupposes the ab-
sence of every established recursive proof relation on which its solution can be
based. The examples given above show that this does not imply that solutions to
philosophical problems are doomed to be arbitrary. There is no scientific philos-
ophy; but equally well, philosophy is a necessary condition for science because
there is no science without foundations.

1.20 REMARK. (I) The examples with applications of mathematical logic to
philosophy given above are mostly concerned with applications to epistemology.
It is more difficult to find applications of mathematical logic to the ontology of
the physical world; but there are indications that recursion theory can be used to
solve ontological problems in the foundations of physics. An example occurs in

(II) It is easier to find examples of applications of mathematical logic to ontol-
ogical problems in the foundations of mathematics. Gödel's incompleteness
theorems are mostly interpreted only epistemologically. There are some math-
ematical insights which cannot be proven by finitistic methods alone, that is, by
methods whose validity cannot be questioned. But Gödel's incompleteness theo-
rems also have ontological implications. These theorems, along with other re-
results in proof theory and recursion theory, give remarkable and surprising in-
sights into the operational structure of mathematics. They give simple and clear
illustrations of how ontology and epistemology are intertwined and cannot be
separated.
1.21 REMARK. The list with applications of concepts and results from mathematical logic to philosophy can be continued indefinitely. Though philosophical logic (for instance modal logic, epistemic logic, deontic logic, dynamic logic) is much less relevant to philosophy than mathematical logic, philosophical logic has nevertheless helped to clarify a few issues in philosophy. (Wittgenstein presumably uses "mathematical logic" in Russell's sense and not in Hilbert's. In that case, "mathematical logic" in his mouth means symbolic logic and includes philosophical logic.) Game theory is indispensable in the study of the foundations of economics, in decision theory, and in the philosophy of action. Several branches of mathematics are useful and necessary in problem solving in the philosophy of science, in philosophy of nature, and in the study of the foundations of physics.

1.22 QUOTATION. "Philosophy must not in any way change the existing linguistic usage; it can after all only describe it. It cannot provide any foundations for it either. It leaves everything as it is.

It also leaves mathematics as it is, and no mathematical discovery can make it progress. A 'main problem in mathematical logic' is for us a mathematical problem like any other." (§124)

1.23 REMARK. Philosophy does not leave everything as it is. Philosophy is foundational research. Revision or amendment of existing foundations has effect on the logical consequences of the foundations, that is on the superstructure, and changes them. New foundations give rise to new superstructures. In the case under consideration, the superstructure is mathematics.

1.24 CONCLUSION. Wittgenstein's assertion that all philosophical problems arise only because of misunderstanding of the correct use of language is ill-
founded. Then his idea of the proper task of philosophy as language therapy is also wrong. Philosophy is foundational studies: the search for truth outside established theories where no established recursive proof relation or verification relation is available. This leaves a vast field of problems for philosophy, including the traditional ontological, epistemological, and ethical problems. Wittgenstein's claim that concepts and results in mathematics and mathematical logic cannot contribute to the development of philosophy is nonsense.

1.25 REMARK. In spite of the shortcomings of Wittgenstein's philosophy, he is still very popular and admired. How is that possible? I think that some of the explanation lies in his ideas about the task of philosophy and about philosophical method. Though no qualified work in philosophy is possible without a mastery of the logical and mathematical methods of philosophy, he promises something else. According to him, mathematics and logic are irrelevant for philosophical thinking. The only competence a philosopher needs is to be able to speak his mother tongue and be able by simple examples to openly expose the rules of language-use, he asserts. Wittgenstein is the lazy philosopher's prophet. A department of philosophy which bans mathematical logic from its own curriculum and research — and almost all philosophy departments in the world do — dooms its own researchers and students to muddle-headedness and intellectual mediocrity.

This point is amply illustrated by the development of twentieth century philosophy. Frege and Whitehead-Russell were pioneers in the development of early modern logic. On the basis of their results in logic, they also made important contributions to philosophy. Taking their work as model, the program of analytical philosophy was formulated. It was supposed that the logic of Frege-Whitehead-Russell — actually a fragment of it — suffices for philosophy. The later development of mathematical logic was believed to be largely irrelevant to philosophy. As a consequence, the work of the leading professional philosophers
became mostly marginal: Moore, the later Russell, Wittgenstein, Carnap, Popper, Quine. And after them came new generations of still weaker philosophers: von Wright, Dummett, Davidson, Kripke, and you name them. The foundational studies of Frege and Whitehead-Russell had degenerated into analytical philosophy. Instead much of the most important progress in philosophy from c. 1925 and on, of which a few examples were given above, were made by mathematicians and mathematical logicians: Hilbert, Brouwer, Heyting, Bishop, Zermelo, Fraenkel, Gödel, P. Cohen, Church, Kleene, Turing, Gentzen, Herbrand, Skolem, Tarski, A. Robinson, P. Lindström, MacLane, Eilenberg, Lawvere, von Neumann, Chaitin, Friedman, Simpson, and many others. The explanation of this phenomenon is that the logicians and mathematicians, in contrast to the professional philosophers, knew the relevant methods and results in logic and mathematics and were able to apply them to philosophical problems. The key to progress in philosophy is to stay close to the development in logic and mathematics and to foundational issues. The key to stagnation and regress in philosophy is to shun logic, mathematics, and foundational issues. These critical remarks on analytical philosophy should not be read as a plea for so-called continental philosophy. Analytical philosophy is mostly superior to continental philosophy. Among analytical philosophers, there is at least a scientific attitude and a striving for exactness. My criticism against analytical philosophy is aimed at the primitive and inadequate logical and mathematical methodology applied and at the too narrow scope of problems considered, asking only about concepts rather than about reality. My critical remarks are a plea for ontological philosophy and foundational studies based on an adequate logical and mathematical methodology: Ask first of all about reality and not only about concepts! Such studies fall outside the scope of both analytical and continental philosophy.

1.26 REMARK. It is of some interest that Gödel's theorem can be used to throw light on Wittgenstein's theory of meaning and the idea of rule following. To un-
derstand arithmetic, one must understand the meaning of the symbols occurring in the language $L(PA)$. We define a *finitary rule* as a program. A program can, in turn, be identified with a Turing machine or a recursive function. I suppose that Wittgenstein, when he claims that language use is governed by rules, by a rule means a finitary rule. The relevant symbols in $L(PA)$ to consider are the non-logical symbols $0, S, +, \cdot, <$, the identity symbol $=$, the connectives $\neg, \land, \lor, \rightarrow, \leftrightarrow$, and the quantifiers $\forall x, \forall y, \exists x, \exists y$, etc. The non-logical symbols and $=$, all represent recursive functions and relations. Therefore their use and meaning are determined by finitary rules. The connectives are defined by finite truth-tables so that even their meanings are governed by finitary rules. We consider the quantifiers. In the general semantics for a predicate logical language, their meanings in a given model $M$ are given by the truth conditions in $M$:

- $M \models \forall x \, B(x) \iff M \models B(a)$ for any $a$ in $M$'s domain
- $M \models \exists x \, B(x) \iff M \models B(a)$ for some $a$ in $M$'s domain

We note that these truth conditions have nothing to say about whether there is any finitary rule in the above sense which determines the meaning of the quantifiers. In the case of $L(PA)$, we are interested in the meaning of the quantifiers in the standard model for PA. Since $\omega$-arithmetic is complete and has the standard model of PA as a model, we can study the truth-conditions for the quantifiers in the standard model by studying $\omega$-arithmetic. We only look at the universal quantifier. The treatment of the existential quantifier is similar. The truth condition is given by the $\omega$-rule:

(26-1) \quad From \, \vdash B(0), \vdash B(1), \vdash B(2), \ldots \, \text{infer} \, \vdash \forall x \, B(x)

and the falsity condition by:

(26-2) \quad From \, \vdash \neg B(n) \text{ for some } n \, \text{infer} \, \vdash \neg \forall x \, B(x)

The falsity condition (26-2) can be derived in the pure predicate calculus for $L(PA)$ and therefore is a finitary rule. Now consider the Gödel sentence $G =$
∀x C(x). It cannot be proven in PA; but all its instances are provable in PA: |

— C(0), |— C(1), |— C(2), .... A condition for using the ω-rule (26-1) in PA, at the object-language level, is that the set of conditions {|

— C(0), |— C(1), |— C(2), ...} can be represented by one sentence. A computer programmed to prove theorems in PA can prove any of C(0), C(1), C(2), ....; but there is no way for it to ascertain the provability of all of C(0), C(1), C(2), ..., because this would imply seeing in PA the set of conditions as an actual infinity. Therefore in PA, at the object-language level, the meaning of the universal quantifier in the sentence ∀x C(x) presupposes the actual infinity. In the proof of Gödel's theorem, at the metalanguage level, all of |

— C(0), |— C(1), |— C(2), ... are shown in a uniform way so that the set of conditions here is a potential infinity. The ω-rule (26-1) in the form

From |— C(0), |— C(1), |— C(2), ... infer |— ∀x C(x)

can now be used. But this presupposes that we see the formal theory PA from the outside. The provability of all the sentences C(0), C(1), C(2), ... can only be ascertained from outside the theory PA. In this case, the meaning of the universal quantifier in ∀x C(x) presupposes that we see PA from the outside. To see PA from the outside means to assume that PA is consistent which in turn is equivalent with the assumption that ∀x C(x) is not provable in PA. (This equivalence is a corollary to the proof of Gödel's second incompleteness theorem.) The unprovability in PA of ∀x C(x) is a necessary condition for our statement that the meaning of the universal quantifier in ∀x C(x) is given by the ω-rule. Therefore the ω-rule gives the meaning in the metalanguage of the universal quantifier in ∀x C(x) just in case PA can be seen from the outside.

We see that the meaning of the universal quantifier in the Gödel sentence ∀x C(x) in PA is given by an infinitary rule and not by a finitary rule. If Wittgenstein's opinion was that all language use and hence all meaning is determined
by finitary rules, then the example shows that he was wrong. We see that the
universal quantifier in \( \forall x C(x) \) in PA gets its meaning from an infinitary rule
which involves an actual infinity. To conceive of an infinite set as an actual in-
finity, we must assume that the set can be seen from the outside as an object and
not only from the inside as a universe. This is the same type of operation as the
one occurring in self-consciousness. It can only be done by a self-conscious be-
ing and in the person's mind. The alternative way to assign meaning to the uni-
versal quantifier in \( \forall x C(x) \) is to see PA from the outside. Even this operation is
of the same type as self-consciousness. This kind of operation, actually a hy-
pothesis, can only be done in the mind. Therefore the meaning of the universal
quantifier is, in the case under consideration, in the speaker's mind and it pre-
supposes consciousness as in my theory of meaning. Then meaning cannot in
general be identified with use and the following of finitary rules. Wittgenstein,
to upkeep his theory of meaning, cannot allow meaning to be determined by in-
finitary rules and must conclude that the universal and existential quantifiers in
some arithmetical contexts have no meaning. Consequently, in Wittgenstein's
philosophy, sentences like \( \text{Consis}_{PA} \) and \( \text{Consis}_{ZF} \) have no arithmetical meaning
— a repugnant conclusion.

To sum up: There are expressions, the use of which in the language is not de-
termined by any finitary rule. Examples are the universal quantifier and the exis-
tential quantifier in arithmetic. This is a weak point in Wittgenstein's idea about
meaning as use, and use in the language as rule following. Instead, the meaning
of these expressions presupposes consciousness and a hypothetical operation
which can only be done in the mind.

Occasionally it seems that Wittgenstein instead expresses himself as a sceptic
about the existence of a definite meaning for a linguistic expression, due to the
inability of rules to convey a meaning. Again the quantifiers in arithmetic can be
adduced as counterexamples. Each of them has a clear and definite meaning,
mastered by anybody who knows the elements of arithmetic. This is so in spite of the fact that there are occurrences of universal quantifiers in arithmetic which do not have truth-conditions determined by any finitary rule. Finitary rules and rule following are not as central and indispensable in meaning attribution as Wittgenstein believed.

1.27 EXAMPLE. (I) Towards the end of Section 2 of "Remarks on Gödel's Philosophy: Private Language and Meaning", I showed that Kripke's argument against the possibility of learning the meaning of words via examples is not valid. Kripke's argument is based on a difficulty about the learning of rules (recursive functions), defined by cases, from a finite number of examples. The crucial point in my objection to Kripke's argument is that definitions by cases in natural languages do not work the way suggested by Kripke. Here I point to another way of showing that Wittgenstein's and Kripke's suspicion about the inadequacy of ostensive definitions for the determination of the meaning of an expression is justified. I use the universal quantifier as an example and get the result as a corollary to Gödel's theorem. This is how Kripke should have proceeded rather than basing his argument on the primitive idea of definition by cases.

(II) One approach to meaning is to say that to know the meaning of an expression is to master a method which determines whether a given use of the expression is correct or not. In the case of the universal quantifier in arithmetic, to know its full meaning is to have a way of proving \( \forall x \ B(x) \) if \( \forall x \ B(x) \) is true and a way of proving \( \exists x \neg B(x) \) if \( \forall x \ B(x) \) is false, for any formula \( B(x) \) with only \( x \) free. Gödel's incompleteness theorem shows that no such method can be learned by examples in the case of the Gödel sentence \( G = \forall x \ C(x) \). To see this, we consider a slight generalisation of the axiom scheme of induction. This generalisation is equivalent with the standard form. Let \( B(x) \) be any formula of \( L(PA) \) hav-
ing only $x$ free. Then the *generalised induction axiom* for $B(x)$ and for any $n \in \mathbb{N}$ is

$$B(0) \land \ldots \land B(n) \land (\forall x \geq n) (B(x) \rightarrow B(S(x))) \rightarrow \forall x \ B(x)$$

This axiom expresses precisely the definition of meaning by examples. $B(0)$, \ldots, $B(n)$ are the examples. The induction step, $(\forall x \geq n) (B(x) \rightarrow B(S(x)))$, represents the etcetera-clause relevant when the rule is clear. Gödel's first incompleteness theorem implies that $\forall x \ C(x)$ cannot be proven in PA. Therefore it cannot be proven by induction, and hence no proof method for $\forall x \ C(x)$ can be learned by examples.

All instances $C(0), C(1), \ldots$ of $\forall x \ C(x)$ are provable in PA. If $K_C$ is the characteristic function of $C(x)$, this implies that all identities

$$(27-1) \quad K_C(0) = 1, K_C(1) = 1, K_C(2) = 1, \ldots$$

are provable in PA. Thus $K_C$ is the constant function which takes the value 1 for all natural numbers as arguments. The trouble with $K_C$ in this context is *not*, as Kripke needs for his argument, that $K_C$ is defined by cases, because $K_C$ is not defined by cases. The trouble with $K_C$ is that no proof method for all the identities in (27-1) can be defined by examples.

If the meaning of the universal quantifier in $\forall x \ C(x)$ cannot be learned by an ostensive definition, how *can* we learn the meaning of the quantifier? In Remark 1.26, I showed that one way of defining the meaning of $\forall x$ in $\forall x \ C(x)$ goes via a proof in the metalanguage of PA. Then we must operate in the metalanguage of PA and assume PA seen from the outside. Another way goes via a use of the $\omega$-rule. Then we must assume the class $\{0, 1, 2, \ldots\}$ seen from the outside as a set. This ability to assume theories, sets, ourselves, and universes seen from the outside is not learned by examples and ostensive definitions. The ability arises in most of us when we are around two years old as a result of a genetic disposition and the interplay with other persons. Once the ability is acquired, it can be
used to give meaning to new expressions, including 'self-consciousness', 'consistency', 'actual infinitity', and the existential and universal quantifiers in arithmetic.

(III) It is possible to give a weaker criterion of meaning than the one used in (II) above. To know the meaning of an expression is to have a criterion which distinguishes between correct and incorrect uses of the expression. In this case, the criterion need not be a method which effectively distinguishes between right and wrong uses of the expression. In the case of the universal quantifier in arithmetic, this criterion could occur in the form of having a definition of \( \forall x \, B(x) \) being true so that \( \forall x \, B(x) \) is used correctly when stated if and only if \( \forall x \, B(x) \) is true, for any formula \( B(x) \) with only \( x \) free. By Tarski's theorem, the truth predicate cannot be defined in any consistent extension of PA. The truth predicate can only be defined in the metalanguage as, for instance, in the Tarski semantics. Thus, in this case, the meaning of the universal quantifier in arithmetic can be defined only if we consider the standard model of PA from the outside and relate it to the language of PA. The general meaning of the universal quantifier in arithmetic contains an implicit reference to self-awareness and cannot be defined ostensively by examples only. Though the meaning of the universal quantifier can be defined ostensively when it ranges over finite sets, this is not necessarily the case when it ranges over infinite sets.

1.28 REMARK. I agree with Wittgenstein in the belief that the meaning of a word or expression can be defined by a rule. The idea that semantic meaning is determined by a rule is older than Wittgenstein. The novelties introduced by him consist in the claims that the rules must be finitary and that each rule associates a word with something external, a set of language-games, and not with something internal in the mind. This conviction is necessitated by the conclusion he draws from the private language argument to the effect that no word or expres-
sion in the common language can refer to something internal and private in the mind. I have shown that no such conclusion can be drawn from this argument. In my philosophy of language, the meaning of a word or expression is almost always determined by a rule which associates the word with something internal in the mind. As shown in Section 2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning", words for colours and pain are associated with phenomenological qualities. These rules are unacceptable from Wittgenstein's point of view because they refer to something inner and private, the phenomenological qualities. In Remarks 1.26-1.27, I have shown how the meanings of the universal quantifier and of the existential quantifier in arithmetic are determined by a rule that associates the quantifier with a state in the mind which is possible only because of the self-awareness of the mind. These rules are unacceptable from Wittgenstein's point of view because they, apart from involving something inner and private, also have a non-finitary component. Because they involve the actual infinity, they have an implicit reference to an oracle, to use an expression introduced by Turing. If we cannot learn the meaning of the quantifiers in arithmetic by examples, how do we learn these meanings? The essential component in the meanings of the quantifiers in arithmetic is our ability of a hypothetical jump out of a universe. This hypothetical jump is essentially the same as the ability to self-consciousness, an ability most human beings acquire gradually at the age of one or two years as a result of a combination of a genetic disposition and the interplay with other human beings. Once this ability is in place, it can be used to give meaning to many terms — like "I", "you" and the quantifiers — which cannot be defined by examples alone and therefore not by language-games alone.

1.29 SUMMARY. A widespread opinion is that Wittgenstein's later philosophy only is a collection of loosely related ideas and remarks. I do not agree with this opinion. The core of the later philosophy is a system with a firm structure and
following a strict logic: (1) The foundations are laid by the private language argument. (2) The theory of meaning as use in the language is a corollary to the private language argument. (3) The therapeutic conception of philosophy follows from the theory of meaning together with a few other premises. This firm underlying structure is then coloured by superposing a wealth of remarks using a pointillist technique. This procedure bestows Wittgenstein's later work with much of the vivid spontaneity of an impressionist painting.

The private language argument, which is fundamental in Wittgenstein's philosophical edifice, also has a simple logical structure. (1) By definition, a private language cannot be used for interpersonal communication. (2) Since ordinary psychological language can be and is used successfully for interpersonal communication, it cannot be private. (3) If psychological terms had referred to something inner and private, they should belong to a private language. (4) Therefore psychological terms do not refer to something inner and private.

A weak point in the foundations of Wittgenstein's later philosophy is its fundamental building block, the private language argument. It is concerned with the philosophy of mind. One of Wittgenstein's purposes with the argument seems to be to disprove psycho-physical dualism. Even for this limited purpose, the argument is invalid as shown in Remark 1.38. There is another philosophy of mind which allows the reference to inner and private states, processes, and events even when we accept Wittgenstein's demand of verifiability. It consists in identifying the mind with the direct awareness of the brain. Mind states are brain states, mind processes are brain processes, and mind events are brain events; they are these states, processes, and events as observed from inside the brain itself. The brain of a self-conscious being is a self-referential system:

(29-1) \[ \text{A person } P's \text{ mind} = \text{P's direct awareness (both act and content) of states, processes, and events in P's own brain.} \]
It is a psycho-physical monism. As shown in detail in sections 1 and 2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning", the private language argument does not exclude this kind of philosophy of mind though Wittgenstein apparently believed so.

Equipped with this psycho-physical monism and with the private language argument disproved, one is not forced to identify meaning with use in the language, as shown in Section 2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning". It can be shown that meaning should be identified with something on the speaker's mind. Ridded of Wittgenstein's theory of meaning, also his therapeutic conception of philosophy fails. Though the correction of conceptual and linguistic misunderstandings may be part of a philosopher's task, it is only a marginal part. Traditional philosophy is still meaningful and an essential part of philosophy. The central issues of philosophy are logic and foundational studies. Foundational studies include the traditional disciplines ontology, epistemology, and ethics. Mostly philosophical work results in theories, theses, and principles — and not only in the disappearance of misunderstandings as Wittgenstein claimed.

1.30 HISTORICAL NOTE. In § 1.11, I said that the central tenets in Hilbert's program are the distinction between real and abstract mathematics and the conservativity result (11-2). I showed how conservativity can be derived from the completeness and finitistically provable consistency of abstract mathematics. Hilbert clearly found such proofs of both completeness and consistency desirable. But it is also clear that Hilbert, at least during the last years before the advent of Gödel's incompleteness theorems, was aware that it suffices with consistency proven by finitistic means in order to infer the conservativity hypothesis (11-2). Completeness is not needed. For Hilbert, consistency is the criterion of existence and hence of truth, no matter whether the theorems are true in a concrete or in an abstract setting. Therefore Hilbert's existence criterion must be
verified and consistency must be finitistically proven for the relevant mathematical theories. Gödel's completeness theorem 1.8-1 verifies Hilbert's existence criterion, albeit by abstract proof methods; but the second incompleteness theorem 1.10(II) shows that the consistency part of the program cannot be realised by finitistic methods only. Thus Gödel's incompleteness theorems undermine Hilbert's program, no matter which method is used to derive conservativity. I now show, under the assumption that the consistency part can be proven finitistically, how a conservativity result can be derived from consistency alone.

LEMMA. Let F be finitistic mathematics and T ⊇ F be an axiomatisable extension of F. Assume that the consistency of T can be proven by finitistic means alone. Then every sentence \( A = \forall x \ C(x) \) of the language of F which is provable in T is also provable in F.

PROOF:

By hypothesis,

(30-1) \( F \vdash \text{Cons}_{T} \)

The following is provable: If \( F \vdash \text{Cons}_{T} \), then

(30-2) \( F \vdash (\text{Thm}_{T}(|A|) \rightarrow A) \)

for any finitistic sentence \( A = \forall x \ C(x) \) where \( C(x) \) represents a recursive relation. The converse is also provable. If (30-2) holds for all \( A = \forall x \ C(x) \), then (30-1) follows. Assume that \( A = \forall x \ C(x) \) can be proven in T:

(30-3) \( T \vdash A. \)

Since T is axiomatisable, (30-3) implies

(30-4) \( F \vdash \text{Thm}_{T}(|A|) \)

Since by hypothesis \( F \vdash \text{Cons}_{T} \), (30-2) is valid. Hence from (30-2) and (30-4),
The result (11-1) — if B is a finitistic theorem of T, then B is true — does not depend on completeness either and can be demonstrated as in § 1.11. Gödel's incompleteness theorem 1.10 shows that the consistency of PA cannot be proven by finitistic methods alone; but it also implies that the conservativity hypothesis (11-2) is false. Consis_{PA} is of the of the finitistic form \( \forall x \ C(x) \). Since it cannot be proven in PA, it cannot be proven by finitistic means alone. On the other hand, it can be proven by abstract methods, for instance in set theory. Therefore Consis_{PA} falsifies the conservativity hypothesis.

1.31 REMARK. The part of Hilbert's philosophy of mathematics which I find valuable and worth preserving is the distinction between real and abstract mathematics. Though, in my opinion, the dividing lines should be drawn in another way than Hilbert did. In particular, I want a distinction between three categories. *Finitary mathematics* contains all recursive functions and computations but no mathematical theory. *Infinitary mathematics* contains all mathematics which can be verified by direct proofs using only finite sets and potentially infinite classes. *Transfinite mathematics* allows both direct and indirect proofs and, along with finite sets and potentially infinite classes, it also allows actually infinite sets. Finitary and infinitary mathematics together coincide with Hilbert's finitistic mathematics. They also coincide with intuitionistic arithmetic. Gödel's theorems show the necessity of transfinite mathematics. It can be shown that the union of finitary and transfinite mathematics is conservative over finitary mathematics. Every computation which can be done by infinitary or transfinite methods can also be done directly by finitary methods alone. Therefore I should prefer to identify abstract mathematics with the union of infinitary and transfinite mathematics. This vindicates to some extent Hilbert's intuition (though not his conjecture). Generally, however, even purely computational problems can often not be
isolated from infinitary and transfinite mathematics. Abstract mathematics provides the models which make it possible to place the computation in its proper physical or mathematical context.

1.32 NOTE. The exposition of Hilbert's program in Remark 1.11 is partly new and partly inspired by Kreisel's and Smorynski's expositions. The presentation is sometimes more stringent than Hilbert's own. The increased stringency is won by using notation and concepts from Gödel's article from 1931 on the incompleteness theorems. The information on Hilbert's program in the historical note 1.30 has profited from a discussion with the Hilbert expert Wilfried Sieg from Carnegie Mellon University. The responsibility for possible errors occurring in the exposition is mine alone.

2. Contradictions: The Wittgenstein-Turing Debate

2.1 INTRODUCTION. During the spring 1939, Wittgenstein gave at the University of Cambridge a series of lectures on the philosophy of mathematics. Alan Turing was allowed to be present. Wittgenstein was concerned with the relationship of mathematics to ordinary everyday language. During one lecture, he considered the relationship between the strict notions of truth and falsity in logic versus the more pragmatic concepts of truth and falsity in ordinary language, including ordinary mathematical language. As an example, he took the presence of a contradiction in a mathematical theory. He claimed that a contradiction in a theory is not as serious as logicians use to say. Pragmatically we might go around the contradiction and isolate it, he claimed, and in this way avoid that it implies every sentence in the language of the theory so that the theory collapses. He was met with opposition from Turing and a discussion ensued between them.
2.2 QUOTATION. "WITTGENSTEIN: … think of the case of the Liar. It is very queer in a way that this would have puzzled anyone — much more extraordinary than you might think. … Because the thing works like this: if a man says 'I am lying' we say that it follows that he is not lying, from which it follows that he is lying and so on. Well, so what? You can go on like that until you are black in the face. Why not? It doesn't matter. … it is just a useless language-game, and why should anybody be excited?

TURING: What puzzles one is that one usually uses a contradiction as a criterion for having done something wrong. But in this case one cannot find anything done wrong.

WITTGENSTEIN: Yes — and more: nothing has been done wrong. … where will the harm come?

TURING: The real harm will not come in unless there is an application, in which a bridge may fall down or something of that sort.

WITTGENSTEIN: … The question is: Why are people afraid of contradictions? It is easy to understand why they should be afraid of contradictions in orders, descriptions, etc., outside mathematics. The question is: Why should they be afraid of contradictions inside mathematics? Turing says, 'Because something may go wrong with the application.' But nothing need go wrong. And if something does go wrong — if the bridge breaks down — then your mistake was of the kind of using a wrong natural law. …

TURING: You cannot be confident about applying your calculus until you know that there is no hidden contradiction in it.

WITTGENSTEIN: There seems to me to be an enormous mistake here. … Suppose I convince Rhees of the paradox of the Liar, and he says, 'I lie, therefore I do not
lie, therefore I lie and I do not lie, therefore we have a contradiction, therefore
2×2 = 369.' Well, we should not call this 'multiplication', that is all. …

TURING: Although you do not know that the bridge will not fall if there are no
contradictions, yet it is almost certain that if there are contradictions it will go
wrong somewhere.

WITTGENSTEIN: But nothing has ever gone wrong that way yet. …"

(* From Andrew Hodges' Alan Turing: The Enigma (1983). The same piece of
conversation is reconstructed in Cora Diamond (editor, 1976), Wittgenstein's
Lectures on the Foundations of Mathematics, 1939. There are a few minor dif-
ferences in the formulations; but in content, the two expositions agree.*)

2.3 REMARK. The general opinion, except possibly among Wittgensteinians, is
that Wittgenstein fell wide of the mark here and that Turing won the debate. I
now advance three observations which show that Turing was, indeed, right.

2.4 THEOREM (Taylor's formula). (I) Let f: \( \mathbb{R} \to \mathbb{R} \) be n times continuously
differentiable in an open interval I around 0. Then for all x ∈ I:

\[
(4-1) \quad f(x) = f(0) + [f'(0)/1!] x + [f''(0)/2!] x^2 + \ldots + [f^{(n-1)}(0)/(n-1)!] x^{n-1} + R_n(x)
\]

(II) If f: \( \mathbb{R} \to \mathbb{R} \) is infinitely many times differentiable in I and if for all x ∈ I
\( R_n(x) \to 0 \) when n → \( \infty \), then

\[
(4-2) \quad f(x) = f(0) + [f'(0)/1!] x + [f''(0)/2!] x^2 + \ldots + [f^{(n)}(0)/n!] x^n + \ldots
\]

2.5 REMARK. Wittgenstein apparently believes that all calculations in applied
mathematics are of the type 2×2 = 4 and that something like 2×2 = 369 always
comes as an immediate consequence of a contradiction in an inconsistent theory
and therefore is easy to recognise. This is a misunderstanding. The usual pattern
of doing calculations in, for instance, engineering is to find a suitable theorem in the mathematical theory and use it to simplify the calculation. A simple example is Taylor's formula. From (4-2), we get for instance:

\[(5-1) \quad e^x = 1 + x + x^2/2! + x^3/3! + \ldots\]

\[(5-2) \quad \sin x = x - x^3/3! + x^5/5! - x^7/7! + \ldots\]

Truncating and using only, e.g., the first four terms, \(e^x\) and \(\sin x\) can be calculated with good approximation in a neighbourhood of 0. Now suppose that the mathematical theory is inconsistent. Then also the following formula can be derived in the theory (because anything can be derived):

\[(5-3) \quad f(x) = f(0) + f'(0) x + f''(0) x^2 + \ldots + f^{(n)}(0) x^n + \ldots\]

This results in the "Taylor series expansions" for \(e^x\) and \(\sin x\):

\[(5-4) \quad e^x = 1 + x + x^2 + x^3 + \ldots\]

\[(5-5) \quad \sin x = x - x^3 + x^5 - x^7 + \ldots\]

If an engineer approximates \(e^x\) and \(\sin x\) by using the formulas (5-4) and (5-5), he might easily get values which deviate sufficiently much from the correct values to result in an unstable bridge, for instance when making a calculation of the strength of material — just as suggested by Turing.

2.6 REMARK. The inference pattern used to derive any sentence from a contradiction is

\[(6-1) \quad \bot \dashv B\]

where \(\bot\) is an arbitrary contradiction, that is, an arbitrary logically false sentence. In the present context, Wittgenstein must be able to block all inferences of the form (6-1). For him to be able to do that, there must be an algorithm which can recognise all contradictions and distinguish them from satisfiable sentences. Wittgenstein apparently assumes that \(\bot\) always is of the form \((A \land \neg A)\).
which *can* be algorithmically recognised. But this is an erroneous assumption. \( \bot \) can be any logically false sentence.

We consider the language \( L = \{0, S, +, \cdot, <\} \) of arithmetic. Let \( K \) be the set of all logically false sentences in \( L \), and let \( T \) be the set of all logically true sentences of \( L \). I show that \( K \) is not decidable. First we define a computable bijection \( f: K \to T \). Let \( A \) be a formula of \( L \). Then \( A = \neg \ldots \neg B \) for some formula \( B \) whose leftmost symbol is not \( \neg \). We call the sequence of negation signs to the left of \( B \) for the *negation matrix* of \( A \). The negation matrix may be empty. Let \( O \) be the set of formulas of \( L \) with an odd number of \( \neg \) in the negation matrix, and let \( E \) be the set of formulas with an even number of \( \neg \) in the negation matrix. If \( A \) has a non-empty negation matrix, then let \( A_R \) denote the result of removing one negation sign from the negation matrix. We can now define the mapping \( f \):

\[
\begin{align*}
  f(A) &= \neg A &\text{if } A \in E \\
  f(A) &= A_R &\text{if } A \in O
\end{align*}
\]

\( f \) is clearly a computable bijection of \( K \) and \( T \). If \( K \) were decidable, then the decision procedure for \( K \) together with the computation method for \( f \) should give a decision method for \( T \). By Church's thesis, \( T \) should be recursive which contradicts Church's theorem 1.15(II). Therefore \( K \) is not decidable and Wittgenstein will not in general be able to recognise whether a derivation of a sentence \( B \) has gone via a contradiction or not.

2.7 **Sequent calculus.** We consider an inconsistent theory \( T \). If \( T \) is inconsistent, then a finite subset \( \Gamma = \{A_1, \ldots, A_n\} \) of the axioms of \( T \) is inconsistent. We can therefore assume that \( \Gamma = \{A_1, \ldots, A_n\} \) contains all the axioms of \( T \). Let the deductive machinery of \( T \) be the sequent calculus \( SK \). We assume that we have the standard formulation of the sequent calculus. The selection of rules from \( SK \) we need here are:
From $\Gamma \vdash \Delta$ infer $\Gamma \vdash \Delta, \Delta$  \hspace{1cm} \text{(Weakening)}

From $\Gamma', A, B, \Gamma'' \vdash \Delta$ infer $\Gamma', B, A, \Gamma'' \vdash \Delta$ \hspace{1cm} \text{(Exchange)}

From $\Gamma, A, A \vdash \Delta$ infer $\Gamma, A \vdash \Delta$ \hspace{1cm} \text{(Contraction)}

From $\Gamma \vdash A, \Delta$ and $\Lambda, A \vdash \Pi$ infer $\Gamma, \Lambda \vdash \Delta, \Pi$ \hspace{1cm} \text{(Cut)}

A deduction is called \textit{cut-free} if it contains no applications of the Cut-rule.

2.8 \textbf{THEOREM} (\textit{Gentzen’s Cut-Elimination theorem}). If there is a deduction in SK of $\Gamma \vdash \Delta$, then there is a cut-free deduction of $\Gamma \vdash \Delta$ in SK.

2.9 \textbf{REMARK}. If $T$ is inconsistent, then any sentence $B$ is provable in $T$. Wittgenstein apparently believes that any such proof must go via a contradiction of the form $(A \land \neg A)$ and use the logically valid inference pattern

$$A, \neg A \vdash B$$

In the sequent calculus, this inference pattern can be proven valid by using the Cut-rule. By hypothesis, $T$ is inconsistent. Therefore we can, for some sentence $A$, prove in SK:

(9-1) \hspace{1cm} $\Gamma \vdash A$

(9-2) \hspace{1cm} $\Gamma, A \vdash$

where (9-2) expresses that $\neg A$ is a consequence of $\Gamma$. From (9-1),

(9-3) \hspace{1cm} $\Gamma \vdash A, B$ \hspace{1cm} \text{(* Weakening *)}

Then from (9-2) and (9-3),

$$\Gamma, \Gamma \vdash B$$ \hspace{1cm} \text{(* Cut *)}

Whence

$$\Gamma \vdash B$$ \hspace{1cm} \text{(* Exchange and Contraction *)}
By Gentzen's cut-elimination theorem, there is a deduction of $\Gamma \vdash B$ in $\text{SK}$ without any use of the Cut-rule. This is then a direct proof of $B$ from the axioms of $T$ without going via any contradiction of the form $(A \land \neg A)$. Therefore the inference pattern used is now

$$(A_1 \land \ldots \land A_n) \vdash B$$

It is easy to see that $(A_1 \land \ldots \land A_n)$ can be any contradiction in the language of $T$. As shown in Remark 2.6, there is no algorithm which in general can decide whether $(A_1 \land \ldots \land A_n)$ is a contradiction or not, and therefore there is no method for blocking the derivation of $B$ from the axioms of $T$; but such a method is what Wittgenstein needs for his idea.

2.10 REMARK. It seems that Turing in the following quotation actually hints at Gentzen's cut-elimination theorem without being explicit. This is perfectly possible since Gentzen's theorem was published in 1934 and must have been known to Turing.

2.11 QUOTATION. The discussion between Turing and Wittgenstein, of which a fragment was quoted in §2.2, ends with the following altercation:

"WITTGENSTEIN: You might get $p \land \neg p$ by means of Frege's system. If you can draw any conclusion you like from it, then that, as far as I can see, is all the trouble you can get into. And I would say, 'Well then, just don't draw any conclusions from a contradiction.'

TURING: But that would not be enough. For if one made that rule, one could get around it and get any conclusion one liked without actually going through the contradiction.

WITTGENSTEIN: Well, we must continue this discussion next time."
2.12 REMARK. Some of Wittgenstein's statements in the debate show a curious ignorance about the nature of mathematical work. He seems erroneously to have believed that all calculations in applied mathematics consist in using simple algorithms like a multiplication algorithm. However, the common way to make a calculation in, for instance, engineering is to get the result as a corollary to a theorem of mathematics. The example in §§ 2.4-2.5 shows calculations as corollaries to the theorem on the Taylor formula. Numerous other examples can be given. One wonders what he did during the years he studied engineering. Apparently, he also believed that foundational systems like Frege's and Cantor's are used in applied mathematics to solve computational problems. This is a misunderstanding. The theories actually used — like arithmetic, algebra, geometry, real and complex analysis, the theory of ordinary and partial differential equations, and functional analysis — are independent of the foundational systems. Moreover, these theories are consistent, and that is why "nothing has ever gone wrong that way yet."

2.13 CONCLUSION. Wittgenstein's idea that contradictions are harmless in ordinary mathematics is untenable. While Wittgenstein's ideas on private language and on meaning at least initially may have some plausibility, his thoughts about the harmlessness of contradictions in mathematical theories are primitive and amateurish.

2.14 EXAMPLE. It might be of some interest that there are examples of theories, namely ω-inconsistent theories, which to some extent are in the neighbourhood of satisfying Wittgenstein's intuitive idea.
An arithmetical theory $T$ is $\omega$-inconsistent if $T$ contains an infinitary configuration of the form

$$T \models A(0), T \models A(1), T \models A(2), \ldots,$$

and $T \models \exists x \neg A(x)$

for some formula $A(x)$. $\omega$-inconsistency is a logically weaker property than inconsistency. Referring to Gödel's Incompleteness Theorem 1.10, it is easy to show that $T = (\text{PA} + \neg \neg G)$ is consistent and $\omega$-inconsistent. It can also be proven that $T = (\text{PA} + \neg G)$ performs all computations correctly in spite of the $\omega$-inconsistency.

2.15 REMARK. Every language, formal or natural, which contains negation also contains contradictions. There is no problem with the possibility of formulating contradictory sentences in a language. Contradictory sentences cannot, however, be accepted in our belief system. They must be removed and replaced by true beliefs. Contradictory theories cannot be tolerated either. They must be replaced by true and hence consistent theories. For the reasons given above, contradictions in a theory cannot be isolated as suggested by Wittgenstein and thus be made harmless.

It is unambiguously clear from the discussion between Wittgenstein and Turing that Wittgenstein was concerned with contradictions in mathematical theories. It might be, however, that Wittgenstein from the outset, in his vague and confused manner, really was aiming at something else. This is indicated by the example he uses in the Quotation 2.2, The Liar Paradox. The Liar sentence, "This sentence is not true", is not just a contradictory sentence. A contradiction has a definite truth value, False. But the Liar sentence has no definite truth value: If it is true then it is false, and if it is false (as a contradiction is) then it is true (which a contradiction is not). Tarski showed that in a consistent theory in a first-order formal language, the Liar sentence cannot be formulated because there is not and cannot be any truth predicate in the theory. In some theories in a higher-
order language or in type theory, it is possible to formulate a hierarchy of typed truth predicates. The typing prevents the formulation of the Liar sentence in such higher order theories. In a formal language, the criterion of which sentences are admissible is purely syntactical. Any sentence which satisfies the formation rules for well-formed formulas is admissible and has a definite truth-value in any model for the language. In a natural language, on the other hand, the syntactical rules do allow the formulation of the Liar sentence, "This sentence is not true". English and most other natural languages contain the possibility of syntactic reference, including self-reference, and it contains one untyped truth predicate as well as negation. That is all we need to formulate the Liar sentence. Therefore the criterion of admissibility of sentences in natural languages must be different from those used for formal languages. We need a criterion of admissibility for natural languages which blocks the Liar sentence.

This problem can be solved by applying the semantic ideas proposed in Section 2, and especially the thesis (2.6-6), of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning": Primarily the meaning of a sentence is something which is on the speaker's mind and which he wants to communicate (like a proposition, thought, perception, belief, intention, wish, query). Therefore a declarative sentence in a natural language to be admissible must express something which can be on a speaker's mind, a proposition. The Liar sentence does not. A sentence is similar to a definite description. A correctly introduced definite description points out a unique individual. Occasionally, a speaker may use a definite description which identifies no individual, like "the present king of France", or several individuals, like "the present member of the French National Assembly". The possibility of such erroneous uses of definite descriptions causes no problems for the use of definite descriptions in general. When they occur and the error is discovered, another expression is normally found to identify uniquely the intended individual. A declarative sentence identifies a proposition, a content of mind, in much the same way as a definite description identifies an
individual. In the ideal case, a sentence represents a unique content of mind, in which case the sentence is an efficient vehicle of communication. Sometimes a sentence may be ambiguous and be satisfied by more than one proposition. In that case, a more precise statement may be needed for successful communication. Some sentences cannot represent any proposition at all. This is the case with the Liar sentence. But just as with definite descriptions without denotation, the possibility of syntactically correct formulations of such sentences does not create any problems for language and communication in general. Just find another sentence which better represents what you have in mind (if anything)! And if you have nothing in mind, don't say anything at all! Thus the criterion of admissibility for sentences in a natural language is not only syntactic but also semantic, and the relevant semantics is the one developed in Section 2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning." The semantic part of the criterion carries greater weight than the syntactic part. The semantic part of the criterion ensures the communicative effectiveness of natural languages. Because of the semantic criterion, the possibility of constructing syntactically correct sentences without any definite truth-value, like the Liar sentence, never makes natural languages collapse as instruments of communication. On the other hand, speakers may, and often do, break syntactical and grammatical rules and still succeed in communicating what is on their minds, that is, the sentence is still semantically effective. A speaker may even sometimes break the semantic rules for one or more words in a sentence and still effectively communicate what is on his mind. An example is Kafka's statement to Max Brod, "If you don't kill me, then you are a murderer", quoted in Remark 2.2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning". In Quotation 2.2 from Wittgenstein in the present section, he says about the Liar sentence: "... it is just a useless language-game." He is right in a way! The Liar sentence is literally a language-game obtained by playing with the syntactic rules of the language and making the victims of the puzzle forget that the primary purpose of a
natural language is to communicate meanings, that is, contents of mind. The Liar sentence is useless for the communication of content of mind; but it has proved to be a useful "language-game" in the understanding of the semantics of languages, natural or formal.

Maybe these reflections are what Wittgenstein really vaguely had in mind in the quoted lecture on the foundations of mathematics from 1939. If they are, he had, at least to a small extent, a sound intuition. If he, on the other hand, was concerned with contradictions in mathematics, as he claims himself, then he was completely wrong.

We may note that even if we strengthen the criterion of admissibility for sentences of a natural language by amending the syntactic and grammatical parts of the criterion with a semantic part demanding that all words and expressions must be used as in ordinary language, this still does not ensure that the sentence has a definite meaning and a definite truth-value. All words and expressions in the Liar sentence are used as in ordinary language. Still it does not have any definite truth-value. This is another way to show that meaning is not use. The Liar paradox is also a counterexample to Wittgenstein's contention that philosophy ought to be language therapy in combination with his theory of meaning as use in the language. All words, expressions, and syntactic constructions in the Liar sentence are used and made according to the semantic and syntactic rules of ordinary language (as Wittgenstein and Turing agree in Quotation 2.2 above). Therefore no language therapy of the sort recommended by Wittgenstein can solve the Liar paradox. Instead the solution to the philosophical problem arising from the Liar paradox must be based on the semantic ideas in Section 2 of "Remarks on Wittgenstein's Philosophy: Private Language and Meaning."

2.16 EPILOGUE. The analyses in the four sections of the two articles show that Wittgenstein had his limitations as a philosopher. This does not in any way im-
ply that he was not a *great* philosopher. Like the arts and entertainment, but in contrast to the sciences, philosophy is based on the star system. The history of philosophy, including contemporary philosophy, is organised around a few stars — "the great philosophers". The fact that much of what the great philosophers have written is nonsense does not affect their status. Similarly some of the leading Hollywood stars are quite mediocre actors. Over many decades, Wittgenstein has proved that he has the star qualities needed to play the role of "great philosopher" on the twentieth century philosophical stage.

2.17 NOTE. The use of Gentzen's Cut-Elimination Theorem in §§2.7-2.9 was suggested to me by the late Thorild Dahlquist.