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IDEAS ON
BELLS THEOREM

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Expositions are given of the Einstein-Podolsky-Rosen argument, Bell's theorem, and the Copenhagen interpretation. It is proved that the assumption of hidden variables is false: Observables cannot all have values simultaneously. It is proved that Bell's inequality depends on an implicit assumption of universality. It is shown that realism and locality are compatible with QM if we drop the operation of universalization. This demands a revision of the foundations of logic. A number of speculative ideas are sketched in the last chapter.

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Strictly speaking my knowledge of physics is insufficient for me to publish on the foundations of physics. But I believe that I have one or two ideas of some interest to communicate. I hope that this will help the reader to overlook the shortcomings in my knowledge of physics.

My primary purpose is to present my own ideas. But I have tried to write in such a way that the booklet can be read also as an introduction to the noble art of Bell's theorem.

Some of the material in Section 4 and in Section 8 was prepared for a course on the philosophy of quantum mechanics which I gave in Uppsala during the autumn semester 1988. I have profited from reactions and comments from the students. Earlier versions of the basic ideas in this booklet were presented in seminars of philosophy in Uppsala during 1986 and 1987. Some of the comments from participants in the seminars have influenced the shaping of this essay.

Among individuals who have been of help I want to mention Paulina Hansen, Sten Lindström, Wlodzimierz Rabinowicz, Sören Stenlund, Stellan Welin, and, particularly, the late Stig Kanger.
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1 PRELIMINARIES

1.1 A large part of this essay will be devoted to the analysis of simple thought experiments. Some are concerned with pairs of protons and the spin attribute. Others are concerned with pairs of photons and the polarization attribute.

1.2 The spin of an elementary particle is its angular momentum in the rest frame of the particle.

By convention the a-component of the spin vector is up or positive if the thumb points in the direction a when the fingers of the right hand curl around the particle and the a-vector in the same sense as the direction of rotation. If the thumb points in the opposite direction of a, the a-spin is down or negative.

1.3 We should not, however, picture the spin as classical rotation for the following reasons. (i) The spin is quantized and therefore discrete. The spin of a proton in a given direction, e.g., can only assume one of two values: \( \pm \hbar/2 \) or \( -\hbar/2 \). Taking \( \hbar/2 \) as a unit of measurement, the two possible values are \( +1 \) and \( -1 \). If the spin of a proton in direction a is \( +1 \ (-1) \), we write a+ (a-).
(i) No matter in what direction we choose to measure the spin of a given proton, the result is always either +1 or -1, and never 0.

1.4 The spin of a particle can be measured by means of Stern-Gerlach magnets. By rotating and setting the Stern-Gerlach device in an arbitrary direction the spin component can be measured in that direction.

1.5 Schematically we represent a spin meter by a box. The direction in which it is set is indicated by an "antenna".

If the spin meter can switch between two or more settings, we indicate this by as many "antennas" as there are settings to switch between.

1.6 The phase space is a 6-dimensional vector space, three base vectors (x, y, z) for the position of a particle and three base vectors (Px, Py, Pz) for its momentum. Two particles occupy the same cell in phase space if the difference between their representative vectors in the phase space is so small that by Heisenberg's indeterminacy relations

\[ \Delta x \Delta P_x \geq \hbar \\
\Delta y \Delta P_y \geq \hbar \\
\Delta z \Delta P_z \geq \hbar \]

it is physically impossible to distinguish them.

1.7 Pauli's exclusion principle. A cell in the phase space can contain at most two protons (or other fermions of the same kind) at the same time. And two protons can occupy the same cell in the phase space only if they have oppositely directed spin vectors.

1.8 By the exclusion principle it is possible to generate pairs of protons with negatively correlated spin.

Two protons PA and PB are forced into the same cell in the phase space in the source S. Alternatively PA and PB may be generated in S in a decay process. They are now in the so-called singlet state. PA and PB are led towards spin meter A resp. B. Then, no matter what common direction a we choose at instruments A and B, we will (practical imperfections aside) get opposite measurement results at A and B.

This type of thought experiment will recur frequently in this essay.

1.9 According to the classical electromagnetic theory of light, light is a wave movement. It consists of an electric field and a magnetic field which are perpendicular to each other and to the direction of the light beam. The fields oscillate many millions of times per second. In Figure 1.9 the light is moving in the direction of the positive x-axis. The electric field is oscillating in the xy-plane. The magnetic field is oscillating in the xz-plane.

1.10 By the photon hypothesis light also behaves particle-like. The square of the wave amplitude in a point is proportional to the probability of finding a photon in the point. This connection is used in the computations in Section 6.
1.11 Light, where the plane of the polarization is constant, is called plane polarized.

The direction, perpendicular to the direction of movement, in which the electric field points, is the polarization direction. In Fig. 1.9, y is the polarization direction.

1.12 Light passing through a calcite crystal is divided into two components.

One of the components is plane polarized parallel to a particular direction in the calcite crystal. The other component is plane polarized in the perpendicular direction.

1.13 Schematically a calcite crystal will be represented by a box with two exit channels.

If the crystal is set to divide the light in horizontally and vertically polarized light, the box will be labelled HV. If we rotate the crystal an angle $\phi$ to the horizontal, it will divide the light in one component polarized in the $\phi$ direction ($\phi^+$) and one component polarized perpendicular to $\phi$ ($\phi^-$). Such a box will be labelled $\mp\phi$.

A calcite crystal can be used to prepare light which is plane polarized in any desired direction. If the crystal is combined with photon detectors (indicated by ']) in the exit channels, it can be used to measure the polarization of each incoming photon.

1.14 One way of indicating a polarization meter and its direction schematically is the one shown in §1.13. An alternative, which will also be used, is to indicate the direction by "antennas" as in §1.5.
1.15 With each photon and each direction \( a \) we associate a polarization variable. If the photon is measured to be polarized in direction \( a \), the value of the variable is set to +1. Otherwise, i.e., if the photon is registered as being polarized in the perpendicular direction, the value is set to -1.

1.15 There are methods of preparing pairs of photons \((P_A, P_B)\) such that \( P_A \) and \( P_B \) have mutually perpendicular polarization no matter what common direction we choose to measure \( P_A \) and \( P_B \) in. Some details can be found in § 4.2. Even such thought experiments will be considered repeatedly in the sequel.

1.16 The following abbreviations will be used frequently.

QM: Quantum mechanics.
SR: The special theory of relativity.
EPR: Einstein, Podolsky, and Rosen; or: the Einstein-Podolsky-Rosen argument.
H: The assumption of hidden variables.
WH: The weak assumption of hidden variables.
L: The locality assumption.
CI: The Copenhagen interpretation.

2 BACKGROUND

2.1 The quantum mechanical state of a system \( Q \) is given by its wave function \( \psi \). Given an observable \( v \) of \( Q \) and a subset \( R \) of \( v \)'s range of possible values we may, by means of \( \psi \), calculate the probability that \( v \in R \).

2.2 Example. The variable \( v \) may, e.g., represent \( Q \)'s position, \( Q \)'s momentum, \( Q \)'s energy, \( Q \)'s spin in a given direction, or \( Q \)'s polarization in a given direction.
(i) If \( \psi(x, y, z) \) is \( Q \)'s wave function and \( \psi^* \) the complex conjugate of \( \psi \), then \( \psi^* \psi \) is the probability density of finding \( Q \) in the point \((x, y, z) \) by measurement.
(ii) Let \( Q \) be a photon prepared so that it is polarized at an angle of 45° to the horizontal. Measure \( Q \) for HV-polarization. It is easy to calculate from the wave function \( \psi \) that the probability that \( Q \) passes through the H-channel is 0.5, and 0.5 that it passes through the V-channel.

2.3 Almost everybody agrees on the above sketched statistical interpretation of the wave function. By means of \( \psi \) one can calculate the expected distribution of measurement results for ensembles of identically prepared particles. The disagreement concerns the interpretation of \( \psi \) as applied to an individual particle.
2.4 According to one opinion \( \Psi \) characterizes the physical state of \( Q \) completely. This idea is the basis of the Copenhagen interpretation. One consequence of this view is that if \( v \) is an observable pertaining to \( Q \) and the value of \( v \) cannot be calculated definitely on the basis of \( \Psi \), then \( v \) has no value before it is measured. According to Bohr and the Copenhagen interpretation it is even meaningless to attribute any value to \( v \) in this situation. In Example 2.2(ii), e.g., the variable \( v_{45} \) for 45° polarization has the value +1 before the measurement. Since the variables \( v_H \) and \( v_H \) for horizontal and vertical polarization cannot be computed from \( \Psi \), \( v_H \) and \( v_V \) have no values before the measurement.

2.5 The alternative is to deny that \( \Psi \) characterizes the physical state of \( Q \) completely. There are, it is claimed, further states in \( Q \) which determine the value of each observable. The parameter \( \lambda \), which describes the physical state of \( Q \), is, of course, in general not observable. Such parameters are called hidden parameters or hidden variables.

In Example 2.2(ii) the hidden variable \( \lambda \) makes sure that \( v_H \) and \( v_V \) have definite values before the moment of measurement — either \( v_H = +1 \) and \( v_V = -1 \), or \( v_H = -1 \) and \( v_V = +1 \). If \( v_H = +1 \), \( Q \) will be registered in the \( H \)-channel. If \( v_H = -1 \), \( Q \) will be registered in the \( V \)-channel.

2.6 Thus the notion of hidden variables depends on two assumptions:
(i) Each quantum system \( Q \) contains hidden states labelled by a parameter \( \lambda \).
(ii) The hidden variable \( \lambda \) determines the result of measuring any observable on \( Q \).

2.7 Remarks. (i) This concept of hidden variables is weaker and more general than the one usually found in the literature. It implies that the hidden variable uniquely determines the values of the observables; but in contrast to the stronger concept it does not imply philosophical determinism.
(ii) In spite of Remark (i) we may generalize the concept of hidden variables by only demanding that \( \lambda \) determines the probability of a certain result at the measurement of an observable on \( Q \). As sketched in Section 5 we can still prove Bell's theorem. But until Section 5 we assume that the observables are uniquely determined by the hidden variable.
(iii) The most important consequence of the assumption of hidden variables is that all observable variables have a definite value simultaneously and at any time. Conversely, given that all observables of \( Q \) have definite values we may identify \( \lambda \) with the set of observable variables; for \( \lambda \) thus defined certainly determines the result of measuring any observable on \( Q \). In the following, and notably in sections 3, 4 and 8, I shall often make this identification.
(iv) The exposition in §§ 2.4-2.5 may give the impression that the Copenhagen interpretation and the hidden variable interpretation form an exhaustive set of alternatives. As we shall see later this is not the case.

2.8 The motives for Bell's work is seen from the diagram.

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  EPR
 /   |
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|\----|------
|     |
|     |
Von Neumann
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|   |
|   |
Bohm
|   |
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|   |
Jauch & Piron
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|   |
|   |
Gleason
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|   |
Bell
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On the one hand there were theorems, one by von Neumann, one by Jauch and Piron, and one by Gleason, which all had as conclusion that hidden variables are incompatible with QM. On the other hand there was a rather strong argument, the Einstein-Podolsky-Rosen argument (EPR), which seemed to im-
ply the existence of hidden variables; and there was Bohm's hidden variable theory which partially duplicated QM.

The purpose of Bell's work was to clarify the relation between QM and hidden variables.

2.9 Bell (1966) analyzed the theorems. He found them mathematically correct; but each one of them relied on a physically dubious hypothesis. These theorems will not be considered further in this essay.

Bohm and others later developed the quantum potential theory (with hidden variables) further so that now it is able exactly to duplicate the predictions of QM. Some comments will be made on this theory, notably in Section 8.

The EPR argument takes us right towards the centre of our main problem. It will be treated in the next section and in Section 10.

3 THE EINSTEIN-PODOLSKY-ROSEN ARGUMENT

3.1 Suppose, as in the Copenhagen interpretation, that the wave function \( \psi \) is a complete characterization of the physical state of a quantum system \( Q \). If at a time \( t_0 \) the value of an observable \( v \) cannot be derived from \( \psi \), then \( v \) has no value at \( t_0 \). If \( u \) is a variable conjugate with \( v \) and \( v \) is measured at a time \( t > t_0 \), then \( u \) has no value at \( t \).

The purpose of the EPR argument is to show the following.
(i) An observable \( v \) may have a sharp value even though it is not computable from \( \psi \). Therefore QM is incomplete.
(ii) Even though the operators corresponding to \( u \) and \( v \) do not commute, \( u \) and \( v \) may both have sharp values at the same time.

3.2 In Einstein, Podolsky and Rosen (1935) position and momentum for pairs of particles are used as example. But the exposition may be simplified if we consider instead the spin components of pairs of half spin particles in two non-collinear directions or the polarization components of pairs of photons in two non-perpendicular directions.

3.3 Experimental arrangement.

Two protons \( p_A \) and \( p_B \) are forced into the same cell in phase space in the source \( S \), where they interact briefly with each other. Since protons are half spin particles, by Pauli's exclusion principle the total spin of \( p_A \) and \( p_B \) is 0, i.e.,
their spin vectors have opposite directions. They are in the singlet state. \( P_A \) is guided towards spin meter A which is set to measure spin components in direction a. \( P_B \) is guided towards spin meter B which also measures in direction a. A and B may be Stern-Gerlach devices.

We also assume that A is closer to S than B is, so that \( P_A \) is measured before \( P_B \) (relative to the laboratory frame). And we assume that \( P_A \) and \( P_B \) are spacelike separated from before measurement A begins until after measurement B is completed.

3.4 In QM, \( P_A \) and \( P_B \) have a common wave function \( \psi \) until one of them is measured. Not until that moment do \( P_A \) and \( P_B \) get their own wave functions \( \psi_A \) and \( \psi_B \).

From \( \psi \) follows

1. \( P(A(a,P_A)=-B(a,P_B)) = 1 \)
2. \( P(A(a,P_A)=+1) = P(A(a,P_A)=-1) = 0.5 \)

before the measurement of \( P_A \)'s spin. Here \( A(a,P_A) \) is the result of measuring the spin of \( P_A \) in direction a; \( B(a,P_B) \) is the result of measuring \( P_B \) in direction a.

From (1) and (2) follows that \( P_A \) and \( P_B \) elicit oppositely correlated spin result, and that it is impossible in QM to attribute any definite spin to \( P_A \) and \( P_B \) before the measurement of one of them.

Equations (1) and (2) are so well verified that they may be considered quantum facts. EPR's strategy is to use these facts to show QM incomplete.

3.5 The EPR argument is based on three assumptions: The completeness principle, the reality principle, and the locality principle. In §§ 3.6-3.9 they are stated and briefly commented upon.

3.6 Completeness Principle (Necessary condition for completeness of a physical theory): Every element of the physical reality must have a counterpart in the physical theory.

3.7 Reality Principle (Sufficient condition for element of reality): If, without in any way disturbing a system, we can predict with certainty, or at least with probability one, the value of a physical quantity at time \( t \), then at time \( t \) there exists an element of physical reality corresponding to this physical quantity.

3.8 Locality Assumption: Elements of reality pertaining to one system cannot be affected by measurements performed at a distance on another system.

3.9 Comments. (i) The idea behind the completeness principle is that a physical theory is incomplete if there is a situation containing an element of reality the existence of which cannot be deduced from the theory together with the set of initial conditions of the situation. It is a strong demand on a theory. But it must be seen in the context of QM, where Bohr claims that the wave function expresses all the reality there is in a quantum system.

(ii) The reality principle seems very reasonable.

(iii) The justification for the locality assumption is the special theory of relativity (SR).

3.10 Argument. In the thought experiment 3.3 we measure \( A(a,P_A) \) first, at time \( t_A \). Say that the result is \( A(a,P_A) = +1 \).

By 3.4(1) we can predict with probability one that \( B(a,P_B) = -1 \) at any time \( t \geq t_A \). This prediction is made on the basis of measurement \( A \) which disturbs \( P_B \) but not \( P_A \). By the reality principle there is for \( t \geq t_A \) an element of reality corresponding to the value -1. Let \( S(a,P_B) \) denote this element of reality, i.e., \( P_B \)'s spin in direction a. Thus \( S(a,P_B) = -1 \). But by the locality principle this element of reality cannot have come into existence as an effect of the measurement performed on \( P_A \). Therefore it must have existed even at times \( t < t_A \). Since it is impossible in QM to assign any
sharp value to \( S(a, p_A) \) before the measurement of \( p_A \), QM is incomplete by the completeness principle.

We have shown Thesis 3.1(i) and the incompleteness of QM.

3.11 Argument. To show Thesis 3.1(ii) we change the experimental arrangement a little.

\[ \text{Let } a \text{ and } b \text{ be two nonparallel directions. Set } A \text{ such that it measures for spin in direction } a \text{ and } B \text{ such that it measures for spin in direction } b. \text{ Now there is a simple way to show that each proton in a pair of protons in the singlet state has a sharp value both in direction } a \text{ and in direction } b. \text{ Measure } p_A \text{ in direction } a \text{ and } p_B \text{ in direction } b. \text{ By locality and results of §§ 3.4 and 3.16 we get:}
\]

\[ S(a, p_A) = \lambda(a, p_A) \]
\[ S(b, p_A) = -\lambda(b, p_B) \]
\[ S(a, p_B) = -\lambda(a, p_A) \]
\[ S(b, p_B) = B(b, p_B) \]

\( S(b, p_A) \) is the value we should have got, had we measured in direction \( b \) instead of direction \( a \). Similarly for \( S(a, p_B) \).

3.12 The conclusion of the EPR argument is that QM is incomplete. In § 3.10 it was shown that a proton's spin variable for an arbitrarily chosen direction \( a \) has a value, independent of measurement. Since the direction \( a \) was arbitrary, we infer, by an accepted logical move, that the spin variables for all directions have values at the same time. Since the argument might as well have treated any other type of observables (position, momentum, polarization), the existence of hidden variables seems to follow (cf. §§ 2.6 and 2.7(iii)).

3.13 In the debate, notably because of Bell's theorem, the validity of the locality assumption has been questioned. Let us use the following abbreviations:

\( L \): The locality assumption.
\( H \): There are hidden variables.

Then, instead of the categorical conclusion

(1) \( H \),

we take the hypothetical

(2) \( L \rightarrow H \)

as conclusion of the discussion.

Note that I have identified the assumption of hidden variables with the assumption that all observables have values simultaneously, as said in Section 2.

3.14 SR lends considerable strength to the locality assumption. The EPR argument makes the assumption of hidden variables look credible. In the next two sections we study consequences of these two assumptions.
4 BELL'S THEOREM

4.1 Bell's theorem rests on the following two assumptions.
**Assumption H**: The existence of hidden variables is assumed. More precisely we assume that all observables have sharp values simultaneously (see §§ 2.6 and 2.7(iii)).
**Assumption L (Locality)**: No influence can propagate with a speed larger than c.

4.2 Experimental arrangement. For the thought experiment leading to Bell's theorem we might consider spin variables for pairs of half spin particles as in Section 3. But for the sake of variation we will instead consider polarization variables for pairs of photons.

An atom of a suitable kind in the light source S is excited. The transition of the atom to the ground state causes the emission of two photons \( P_A \) and \( P_B \) in rapid succession and in opposite directions.

A and B are polarization meters which can be set to measure the polarization of a photon in three directions a, b, c, all perpendicular to the axis AB. We assume that a random or deterministic mechanism chooses the direction of the setting of A and of B only after the photons \( P_A \) and \( P_B \) have left the source S. Then the polarization of \( P_A \), the setting of A and the measurement of \( P_A \) will be spacelike separated from the setting of B and the outcome of the measurement of \( P_B \), and vice versa.

Included in A and B are also photon multipliers which register for the given setting whether the photon chooses the + channel or the - channel.

4.3 The most important property of the photon pair \( (P_A, P_B) \) generated as described in § 4.2 is that they get oppositely correlated polarization. I.e., if A and B are set to measure in the same direction, e.g., direction a, then we will get either \( A(a, P_A) = +1 \) and \( B(a, P_B) = -1 \) or \( A(a, P_A) = -1 \) and \( B(a, P_B) = +1 \).

This correlation can be derived in QM. It can also be verified experimentally. We therefore consider it a fact and use it freely in the proof.

4.4 In this section we make the following two assumptions.
(i) We assume that the polarization values of a, b, c are deterministic functions of \( \lambda \). (This is implicit in Assumption H.)
(ii) We assume that all of the polarization variables a, b, c in \( P_A \) and in \( P_B \) have a constant value from the moment they leave S until one of them is measured.

In Section 5 it will be shown that (ii) can be derived from the correlation mentioned § 4.3, which in turn is a quantum fact. Thus (ii) is, as an assumption, really redundant.

4.5 Corollary. (i) As a consequence of 4.4(i) the polarization variables a, b, c have sharp and definite values at any time as explained in § 2.7(iii).
(ii) Statement 4.4(ii) may be derived from the correlation of § 4.3 and Assumption L. Statement 4.4(ii) and one more application of locality give the following strengthening of 4.4(ii): All the polarization variables a, b, c in \( P_A \) have constant values between S and A. A similar consequence holds for \( P_B \).
4.6 The setting mechanism for A chooses each of the settings a, b, c with probability 1/3, and similarly for B. Therefore there will be 9 different possible settings for the combined apparatus (A,B): (ab), (ac), (bc), (ba), (bb), (ba), (ca), (cb), (cc). Each comes up with probability 1/9.

4.7 We get the following picture of what happens in an experiment. A sequence of N pairs of photons (P_A,P_B) is generated in S and with such intervals that only one pair is in the interval at the same time. A and B are set to direction a, b or c after P_A and P_B have left S but before they reach A resp. B.

With each photon leaving S is associated a set of polarization values (+ or -) for a, b, c. Therefore for the whole experiment we get two 3-column lists with N lines. Each line will be categorized according to the setting. For instance:

<table>
<thead>
<tr>
<th>Setting</th>
<th>P_A</th>
<th>P_B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ab)</td>
<td>a</td>
<td>+</td>
</tr>
<tr>
<td>(cc)</td>
<td>b</td>
<td>-</td>
</tr>
<tr>
<td>(ab)</td>
<td>c</td>
<td>-</td>
</tr>
<tr>
<td>(bc)</td>
<td>b</td>
<td>+</td>
</tr>
<tr>
<td>(ba)</td>
<td>b</td>
<td>-</td>
</tr>
<tr>
<td>(cb)</td>
<td>a</td>
<td>+</td>
</tr>
<tr>
<td>(aa)</td>
<td>c</td>
<td>-</td>
</tr>
<tr>
<td>(ac)</td>
<td>c</td>
<td>+</td>
</tr>
</tbody>
</table>

By Assumption H and 4.5(1) the polarization variables a, b, c have definite values. By 4.5(II) these values are constant for each photon until the moment of measurement for the photon. By § 4.3 P_A's and P_B's values will be oppositely correlated in all directions.

As an example consider the first line in the table. A measures P_A in direction a with result +1. B measures P_B in direction b with result -1. Because of the correlation, the measurement at B of P_B in direction b is an indirect measurement of the value of P_A's polarization variable in direction b. The polarization value in the third direction, here c, is impossible to inspect. We assume nevertheless, in accordance with the assumption of hidden variables, that each photon has a polarization value even in the third direction.

4.8 We first prove a result about single lists with three columns and N lines like, e.g.,

```
+ - - +
- + + -
+ - - +
- + + -
- + - +
- + - +
+ + + -
```

Thus at first we look only at the polarization of single photons.

Let \( L(a=+,b=-) \) be the number of lines in the sample having + in column a and - in column b. Let \( L(a=+,b=+,c=+) \) denote the number of lines having +, +, + in columns a, b, c respectively; etc.

4.9 Lemma. For all lists as defined in § 4.8,

\[
L(a=+,c=+) \leq L(a=+,b=+) + L(b=+,c=+)
\]

Proof:

Since each line has in column b either + or -, we have

1. \( L(a=+,c=+) = L(a=+,b=+,c=+) + L(a=+,b=+,c=-) \)

Analogously,

2. \( L(a=+,b=+) = L(a=+,b=+,c=+) + L(a=+,b=+,c=-) \)

3. \( L(b=-,c=+) = L(a=+,b=-,c=+) + L(a=+,b=-,c=-) \)

Add (2) and (3)

4. \( L(a=+,b=+) + L(b=-,c=+) = L(a=+,b=+,c=+) + L(a=+,b=+,c=-) + L(a=+,b=-,c=+) + L(a=+,b=-,c=-) \)

We see that the first and third term on the right-hand side of (4) are the same as the terms on the right-hand side of (1). Since the value of \( L \) is always \( \geq 0 \), we get from (1) and (4)

\[
L(a=+,c=+) \leq L(a=+,b=+) + L(b=+,c=+)
\]

4.10 The inequality of Lemma 4.8 is almost Bell's inequality. The trouble with it is that \( L \) is a function of hidden
variables and therefore not measurable. We now replace $L$ with a measurable function $M$.

We consider again pairs $(p_A, p_B)$ of photons and the 9 possible settings of $(A, B)$. Let

$M(a+b+) = \text{the number of pairs of photons in the sample such that } p_A \text{ is measured in direction } a \text{ with result } +, \text{ and } p_B \text{ is measured in direction } b \text{ with result } +.$

$M(c-a+) = \text{the number of pairs of photons such that } p_A \text{ is measured in direction } c \text{ with result } -, \text{ and } p_B \text{ is measured in direction } a \text{ with result } +.$

Etc.

Since $M$ refers only to the macroscopic settings of $A$ and $B$ and to instrument reactions, $M$ is measurable.

To relate $L$ and $M$ we use the observation of § 4.7 that though a photon, say $p_B$, cannot be measured directly in two different directions, say $a$ and $b$, we may apply the following indirect procedure:

Measure $p_B$ in one direction, e.g., direction $a$. Measure $p_A$ in direction $b$. Infer the polarization of $p_B$ in direction $b$ by the anti-correlation of § 4.3.

Lemma 4.11 will be based on probability theory. $X = Y$ will mean that $X$ and $Y$ have very probably nearly the same value. I do not calculate margins of error and standard deviations for the following reasons:

(i) We know that it can be done.

(ii) We do not need them for the conceptual analysis of the proof of Bell's theorem.

Let $L$ be defined on the list of polarization values for the $p_B$. Thus, e.g., $L(a=+, b=-)$ denotes the number of $p_B$ among the $N$ pairs which have $+$ in direction $a$ and $-$ in direction $b$.

4.11 Lemma. For all sufficiently large numbers $N$ of pairs of photons

(1) $M(a-c+) = L(a=+, c=+)/9$
(2) $M(a-b+) = L(a=+, b=+)/9$
(3) $M(b+c+) = L(b=-, c=+)/9$

Proof:
A pair is by the measurement registered as an $(a-c+)$-pair iff

(i) $p_A$ and $p_B$ have the following values

\[ \begin{array}{ccc}
   a & b & c \\
   - & + & + \\
\end{array} \]

(ii) $a$ is set to direction $a$ and $B$ to direction $c$.

The number of $p_B$-photons satisfying condition (i) is $L(a=+, c=+)$. For any pair $(p_A, p_B)$ of photons, the probability that $p_A$ is measured in direction $a$ and $p_B$ in direction $c$ is $1/9$. Therefore

$M(a-c+) = L(a=+, c=+)/9.$

Equations (2) and (3) are proved similarly.

4.12 Remark. For the proof of Lemma 4.11, Corollary 4.5(ii) is essential. If the measurement of one photon could cause an immediate change in the polarization of its twin photon, the statistical relations 4.11(1)-(3) may no longer hold.

Since Corollary 4.5(ii) depends heavily on the locality assumption, so does Lemma 4.11.

4.13 Theorem. $M(a-c+) \leq M(a-b+) + M(b+c+)$

Proof:
By Lemma 4.9,

(1) $L(a=+, c=+) \leq L(a=+, b=+) + L(b=-, c=+)$

By (1) and equations (1)-(3) of Lemma 4.11 we get

$9M(a-c+) \leq 9M(a-b+) + 9M(b+c+)$

Hence

$M(a-c+) \leq M(a-b+) + M(b+c+).$

4.14 Inequality 4.13 is essentially Wigner's form of Bell's inequality. It is, in principle, empirically testable. Thus, amazingly, we have from an apparently purely ontological assumption without any physical content (namely Assumption H) deduced a physical conclusion.
4.15 The proof of Bell's theorem given here is by no means the shortest and most elegant. But it is very concrete and transparent; and it is, I think, the proof which is most easily amenable to analysis.

4.16 From the proof given in this section it is clear that Bell's inequality only depends on the assumptions of hidden variables and locality.

Assumption H is used to associate two three column lists of polarization values with the sample of pairs of photons.

Assumption L is used to make sure that the polarization values for a photon are constant from the photon leaves S until the moment when it is measured.

5 A GENERALIZED VERSION OF BELL'S INEQUALITY

5.1 In this section I outline a proof of a generalized Bell theorem. The proof is compared with the one given in Section 4. I show why the proof of Section 4 is better suited for our purposes in this essay.

5.2 Experimental arrangement.

Pairs of protons are generated in S in the singlet state.
(The use of protons is not essential. We may use photons or other particles as well.) A and B are spin meters. A can be set to measure in directions a and a'. B can be set to measure in directions b and b'. We assume that A and B are set only after p_A and p_B have left S. We assume that the setting of A, the measurement of p_A, and the deflection in A are spacelike separated from the setting of B, the measurement of p_B, and from the deflection in B, and vice versa.

5.3 As in Section 4 we assume the existence of hidden variables, though in the weak form of Assumption WH, and locality (Assumption L). The hidden variables assumption is weakened in accordance with 5.4(iii) below.

5.4 The important differences compared with Section 4 are the following.
(i) We consider four directions $a$, $a'$, $b$, $b'$, not three. Some of the directions may or may not coincide. Each spinmeter is set in one of two directions, not one of three as in Section 4.

(ii) We no longer presuppose perfect anti-correlation between $p_a$ and $p_b$'s spin for the same direction of measurement.

(iii) We no longer presuppose that the measurement results in $\lambda$ and $B$ are deterministic functions of $p_a$ and $p_b$'s hidden variables. We only assume that the hidden parameter $\lambda$ determines the probability for a given measurement result. Therefore in this section we cannot identify the hidden variables with the set of all observables.

(iv) We no longer presuppose that the spin variables of $p_a$ and $p_b$ have a constant value from they leave $S$ until one of them is measured.

5.5 The proof is adapted from d'Espagnat (1984) where details can be found; I only give the main points of the proof.

5.6 The variables corresponding to the hidden states are represented by $\lambda$. It does not matter whether $\lambda$ denotes a single real number, a set of real numbers or a set of functions, and whether the variables are continuous or discrete. Here I will treat $\lambda$ as one single continuous real variable. The integrals will therefore be the usual Riemann integrals.

5.7 Bell's Theorem. Assume the existence of hidden variables (WH) and locality (L). Then

$$|C(a,b) + C(a',b')| + |C(a',b) - C(a,b')| \leq 2$$

Proof:

$C$ is a correlation function and defined below.

We use $A$ in the proof to represent the deflection of instrument $A$ by the measurement of $p_a$. Similarly for $B$. Then

$$A = \pm 1 \quad B = \pm 1.$$  

$P(A|\lambda,a,b,B)$ will denote the probability of result $A$ by the measurement of $p_a$ given that (i) the hidden variable has the value $\lambda$, (ii) instrument $A$ is set to $a$ and instrument $B$ to $b$, and (iii) $B$ is the result of the measurement of $p_b$.

$P(A|\lambda,a,b) P(B|\lambda,a,b)$ will denote the probability of result $A$ by the measurement of $p_a$ and result $B$ by the measurement of $p_b$ given that (i) the hidden variable has the value $\lambda$, and (ii) instrument $A$ is set to $a$ and instrument $B$ to $b$.

The rule for conditional probability gives

$$P(A|\lambda,a,b,B) = \frac{P(A|\lambda,a,b) P(B|\lambda,a,b)}{P(B|\lambda,a,b)}$$

It may be rewritten as

$$P(A|\lambda,a,b,B) = P(A|\lambda,a,b) \cdot P(B|\lambda,a,b)$$

By the locality assumption the result $A$ of the measurement of $p_a$ is independent of the setting $b$ of instrument $B$ and of the outcome $B$ of the measurement of $p_b$. Therefore

$$P(A|\lambda,a,b,B) = P(A|\lambda,a)$$

Similarly,

$$P(B|\lambda,a,b) = P(B|\lambda,b)$$

Insertion in (3) gives

$$P(A|\lambda,a,b) = P(A|\lambda,a) P(B|\lambda,b)$$

We now define the expected mean value $C(\lambda,a,b)$ of the product $A \cdot B$ for a pair of particles $(p_a, p_b)$ with hidden state $\lambda$ given that instrument $A$ is set to $a$ and instrument $B$ to $b$.

$$C(\lambda,a,b) = \sum_{A,B} A \cdot B \cdot P(A,B|\lambda,a,b)$$

(Written in out in full the definition looks as follows:

$$C(\lambda,a,b) = P(1,1|\lambda,a,b) - P(-1,1|\lambda,a,b)$$

$$- P(-1,-1|\lambda,a,b) + P(1,-1|\lambda,a,b).$$)

Clearly by the triangle inequality,

$$|C(\lambda,a,b)| \leq 1.$$  

From (4) and (5) we obtain after some algebraic manipulations

$$|C(\lambda,a,b) + C(\lambda,a',b')| + |C(\lambda,a',b) - C(\lambda,a,b')| \leq 2$$

This is almost Bell's inequality (in generalized form). One problem is that $C(\lambda,a,b)$ is not measurable, because $\lambda$ is
a hidden variable. We therefore define the correlation coefficient \( C(a,b) \) as the mean value of all the \( C(\lambda,a,b) \).

\[
(8) \quad C(a,b) = \int C(\lambda,a,b) \, p(\lambda) \, d\lambda
\]

\( p(\lambda) \) is the probability density for the state \( \lambda \) so that

\[
(9) \quad \int p(\lambda) \, d\lambda = 1
\]

Thus \( C(a,b) \) is the expected mean value of the product \( A \cdot B \) for an arbitrary pair \((p_A,p_B)\) when instrument \( A \) is set to \( a \) and instrument \( B \) to \( b \). \( C(a,b) \) is measurable via the connection between probability and relative frequency.

If we multiply both sides of (7) by \( p(\lambda) \), integrate with respect to \( \lambda \), and use the triangle inequality, we obtain

\[
(10) \quad |C(a,b) + C(a,b')| + |C(a',b) - C(a',b')| \leq 2
\]

5.8 The conclusion of Theorem 5.7 is a generalized form of Bell's inequality. From this inequality it is easy to extract Bell's original inequality as a corollary.

5.9 Corollary. Assume WH and L, and strict correlation. If \( b' = a \), then

\[
|C(a',b) - C(a',b')| \leq 1 + C(a,b)
\]

**Proof:**

Set instrument \( A \) and \( B \) such that \( b' \) and \( a \) are parallel.

Then, because of the strict correlation,

\[
(1) \quad C(a,b') = -1
\]

From (1) and the generalized inequality

\[
(2) \quad |C(a,b) - 1| + |C(a',b) - C(a',b')| \leq 2
\]

But

\[
|C(a,b) - 1| = 1 - C(a,b)
\]

Therefore

\[
|C(a',b) - C(a',b')| \leq 1 + C(a,b)
\]

5.10 In Section 4 and in § 5.7 two different proofs of Bell's theorem have been given. In §§ 5.11-5.14 the two proofs will be compared.

5.11 In the proof given in Section 4 each instrument could choose between three settings. In the proof given in this section the instruments could switch between only two settings. In practical experiment, where we want the setting of the two instruments \( A \) and \( B \) to be spacelike separated, it must be possible to change the setting of an instrument in a few nanoseconds. This is easier to accomplish if there are only two settings to switch between. But if we inspect Inequality 4.13 we see that to test it instrument \( A \) needs switch only between the settings \( a \) and \( b \) and instrument \( B \) only between \( b \) and \( c \). Moreover, the problem of testing Bell's inequality experimentally is settled (see Section 6). Our main purpose is the theoretical analysis of Bell's theorem. For this purpose a version of the theorem, which is as simple, concrete, and transparent as possible, is desirable.

The beneficial consequences of considering three distinct directions \( a, b, c \) for each instrument rather than two are the following:

(i) We can identify the set of hidden states with the set of possible ways to attribute values to the observables. In the given connection (assuming also strict correlation) there are essentially only eight distinct states to consider:

<table>
<thead>
<tr>
<th>( \text{State} )</th>
<th>( p_A )</th>
<th>( p_B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>+ + +</td>
<td>+ - +</td>
</tr>
<tr>
<td>2</td>
<td>+ + -</td>
<td>- - +</td>
</tr>
<tr>
<td>3</td>
<td>+ - -</td>
<td>+ + -</td>
</tr>
<tr>
<td>4</td>
<td>+ - +</td>
<td>- + +</td>
</tr>
<tr>
<td>5</td>
<td>- + +</td>
<td>+ - -</td>
</tr>
<tr>
<td>6</td>
<td>- + -</td>
<td>+ + -</td>
</tr>
<tr>
<td>7</td>
<td>- - +</td>
<td>+ - +</td>
</tr>
<tr>
<td>8</td>
<td>- - -</td>
<td>+ + +</td>
</tr>
</tbody>
</table>

An appreciable simplification results.

(ii) We avoid integrals. Ordinary finitesimal sums suffice. Again we gain in transparence.

5.12 In Section 4, but not in the present section, we assumed perfect negative correlation. The assumption of perfect negative correlation is said to be "experimentally unrealistic" (Clauser, Horne, Shimony, Holt (1969)). This can safely be attributed to practical imperfections in the ex-
perimental equipment. Perfect negative correlation in experiments 4.2 and 5.2 can be proved in QM and demonstrated with sufficient accuracy experimentally for it to be considered a fact. Since our purpose is purely theoretical, we need not worry about practical limitations.

5.13 In Section 4, but not in the present section, we assumed that the values of the observables (polarization or spin) are deterministic functions of the hidden variable \( \lambda \). In § 5.12 I showed that for theoretical purposes we may safely assume perfect negative correlation. Here I will show that this correlation implies determinism (given the locality assumption). Intuitively this is clear. If there is the least probabilistic looseness as regards which value a spin variable should assume by measurement, we cannot always get oppositely correlated values in the same direction. But a more formal proof can be given:

\[
0 = P(A=1 \wedge B=1|\lambda,a,a) \quad \text{(perfect correlation)}
\]

\[
0 = P(A=1|\lambda,a) \cdot P(B=1|\lambda,a) \quad \text{(see 5.7(4))}
\]

\[
0 = P(A=-1 \wedge B=-1|\lambda,a,a) \quad \text{(perfect correlation)}
\]

\[
P(A=-1|\lambda,a) \cdot P(B=-1|\lambda,a) \quad \text{(see 5.7(4))}
\]

From (1) and (2) we get the equation system

\[
P(A=1|\lambda,a) + P(B=1|\lambda,a) = 1
\]

\[
P(A=1|\lambda,a) \cdot P(B=1|\lambda,a) = 0
\]

which has only the following two solutions:

(i) \( P(A=1|\lambda,a) = 1 \) \& \( P(B=1|\lambda,a) = 0 \)

(ii) \( P(A=1|\lambda,a) = 0 \) \& \( P(B=1|\lambda,a) = 1 \)

Thus the outcome of a measurement is completely determined by \( \lambda \).

5.14 In Section 4, but not in the present section, we assumed that the values of the observables of \( p_A \) and \( p_B \) were constant from they leave \( S \) until one of them was measured. Again this assumption may be seen to follow from the existence of perfect negative correlation (given locality).

For suppose that the spin of \( p_A \) and \( p_B \) changes randomly (in a subjective sense) with time though such that (i) the spin is always determined completely by \( \lambda \) (which may be identified with the initial conditions in \( S \)), and (ii) \( p_A \) and \( p_B \) have at every moment negatively correlated spin. Let us suppose that \( p_A \) is measured first, in direction \( a \). By locality, the variations in \( p_B \)'s spin in direction \( a \) are independent of the measurement performed on \( p_A \). But then the measurement of \( p_B \)'s a-spin cannot always give the opposite value of \( p_A \)'s a-spin.

5.15 From §§ 5.11-5.14 we see that the extra assumptions made in Section 4 do not weaken the result proved there compared with the proof given in the present section. Since the proof of Section 4 is simpler and more transparent, I will mainly use that proof for analysis in the following sections.
6.1 In this section we show that Bell's inequality is incompatible with QM. We first consider the form derived in Section 4
\[ M(a-c+) \leq M(a-b+) + M(b+c+) \]
N denotes the number of pairs of photons measured in Experiment 4.2. \( V_{ab} \) denotes the angle between the vectors \( a \) and \( b \).

6.2 Lemma. In QM we have

1. \[ M(a-c+) = (N/18) \cos^2 V_{ac} \]
2. \[ M(a-b+) = (N/18) \cos^2 V_{ab} \]
3. \[ M(b+c+) = (N/18) \sin^2 V_{bc} \]

Proof:
A pair \((p_A, p_B)\) is counted as an \((a-c+)\)-pair iff the following three conditions are satisfied.
(i) Instrument A is set to direction a and instrument B to direction c. The probability of this is 1/9.
(ii) The measurement of \( p_A \) must give a result -1. The probability of this is 1/2.
(iii) Given that \( p_A \) has polarization a-, \( p_B \) must have c+.
The probability of this is calculated as follows. If \( p_A \) has a-, \( p_B \) has a+.

From the figure we see that since the probability is proportional to the square of the c-component of the electric field of an a+-polarized wave, this probability is \( \cos^2 V_{ac} \).
Hence
\[ M(a-c+) = (1/9) (1/2) N \cos^2 V_{ac} = (N/18) \cos^2 V_{ac}. \]

We consider (3). Since \( p_A \) is \( b+ \), \( p_B \) is \( b- \). Then, as seen from the figure, the probability that \( p_B \) is \( c+ \) is \( \cos^2(90-V_{bc}) = \sin^2 V_{bc} \).
Thus
\[ M(b+c+) = (N/18) \sin^2 V_{bc}. \]

6.3 Theorem. Assume QM. Then there are directions a, b, c such that
\[ M(a-c+) > M(a-b+) + M(b+c+) \]

Proof:
Suppose there are no such a, b, c. Then Bell's inequality
\[ M(a-c+) \leq M(a-b+) + M(b+c+) \]
is consistent with QM. Applying the values from Lemma 6.2 to inequality (1) we get
\[ (N/18) \cos^2 V_{ac} \leq (N/18) \cos^2 V_{ab} + (N/18) \sin^2 V_{bc} \]
which simplifies to
\[ \cos^2 V_{ac} \leq \cos^2 V_{ab} + \sin^2 V_{bc} \]
Choose, e.g., \( a, b, c \) such that
\[
V_{ab} = 60^\circ \\
V_{ac} = 20^\circ \\
V_{bc} = 40^\circ
\]
Then by (2)
\[
\cos^2 20 \leq \cos^2 60 + \sin^2 40
\]
i.e.,
\[
0.8830 \leq 0.25 + 0.4132 \\
\leq 0.6632
\]
which is false. In fact, every choice of directions \( a, b, c \) such that
\[
0 < V_{ac} < 30^\circ \\
V_{ab} = 3 V_{ac}
\]
yields a contradiction.

6.4 Even Bell’s original inequality
(1) \[|C(a',b) - C(a',b')| \leq 1 + C(a,b)\]
(where \( b' = a \)) and the generalized Bell inequality
(2) \[|C(a,b) + C(a,b')| + |C(a',b) - C(a',b')| \leq 2\]
are incompatible with QM.

By an argument akin to the one of § 6.2 it is shown in QM that
(3) \[C(a,b) = -\cos V_{ab}\]
To show (1) incompatible with QM a suitable choice of angles may be
\[
V_{a'b'} = V_{ab} = 60^\circ \\
V_{a'b} = 120^\circ
\]
For then from (1) and (3)
\[
| -\cos V_{a'b} + \cos V_{a'b'} | \leq 1 - \cos V_{ab}
\]
Thus
\[| -\cos 120 + \cos 60 | \leq 1 - \cos 60 \]
i.e.,
\[1 = |0.5 + 0.5| \leq 1 - 0.5 = 0.5 \]
which is false.

To show (2) incompatible with QM a suitable choice of angles may be
\[
V_{ab} = 225^\circ \\
V_{ab'} = 135^\circ \\
V_{a'b} = 135^\circ \\
V_{a'b'} = 45^\circ
\]
For then from (2) and (3)
\[
| -\cos V_{ab} - \cos V_{ab'} | + | -\cos V_{a'b} + \cos V_{a'b'} | \leq 2
\]
Thus
\[ |-\cos 225 - \cos 135| + |-\cos 135 + \cos 45| \leq 2 \]
and hence
\[
2.8284 = |0.7071 + 0.7071| + |0.7071 + 0.7071| \\
\leq 2
\]
which is false.

6.5 One consequence of the argument so far is that no local hidden variable theory is able to duplicate exactly all the predictions of QM. The argument does not decide whether the QM inequality or Bell's inequality is true.

A number of experiments to answer this question have been performed. Experiments by Aspect, Dalibard, Grangier, and Roger 1981 and 1982 are generally considered to be conclusive. They used equipment which worked so fast that the setting and measurement events at instrument A actually were spacelike separated from the setting and measurement events at instrument B. The experiments confirmed QM and falsified Bell's inequality.

6.6 Problem. The results in §§ 6.4–6.5 raise a problem: Bell's inequality was derived from the assumptions of hidden variables (H) and locality (L). Since Bell's inequality is false, at least one of H and L must be false. Which one?

This is a main problem in this essay. In Section 3, reasons were given for assumptions H and L. The difficulty of the problem is proportional to the strength of these arguments.

6.7 We make a logical analysis of the argument so far. Let us use the following parameters:
H: The assumption of hidden variables.
L: The locality assumption.
B: Bell's inequality.

Then
\[
(1) \quad L \to H \\
(2) \quad H \land L \to B \\
(3) \quad \neg B
\]

Thus EPR together with Bell's theorem leads us to the amazing conclusion that the locality assumption is false.

\[
(4) \quad \neg (H \land L) \\
(5) \quad \neg L
\]

(1), (4), logic
(2), (3), logic
7 UNIVERSALITY IN BELL’S INEQUALITY

7.1 In this section we make a first step towards a solution of Problem 6.6. In § 6.7 we concluded that the locality assumption is false. This seems to solve Problem 6.6. But non-locality is hard to accept; and we have certainly not worked our way to the bottom of the problem yet.

7.2 We consider the Bell inequality
   (1) \( M(a-c^+) \leq M(a-b^+) + M(b+c^+) \)

Let
   \( B(a, b, c) \)

denote this relation. Then the conclusion of Bell's theorem is

   \( (\forall a)(\forall b)(\forall c) \) \( B(a, b, c) \),

i.e., Bell's inequality is satisfied by all directions \( a, b, c \).

In QM we have on the other hand
   \( (\exists a)(\exists b)(\exists c) \) \( B(a, b, c) \)

\( (\exists a)(\exists b)(\exists c) \) \( \neg B(a, b, c) \)

i.e., some choices of directions \( a, b, c \) satisfy Bell's inequality while others falsify the same inequality.

We see that the difference between QM and the conclusion of Bell's theorem is the following: The latter claims Bell's inequality to be universally valid, while according to QM it is only of limited validity.

7.3 Example. First we rewrite Bell's inequality in the form
   (1) \( M(a-b^+) + M(b+c^+) - M(a-c^+) \leq 0 \)

By the identities of Lemma 6.2 we get that (1) in QM is equivalent to
   (2) \( \cos^2V_{ab} + \sin^2V_{bc} - \cos^2V_{ac} \geq 0 \)

We consider here only choices \( a, b, c \) which satisfy the relations
   (3) \( V_{bc} = 2V_{ac}, \quad V_{ab} = 3V_{ac} \)

Using (3) and writing \( x \) for \( V_{ac} \), the left-hand side of (2) may be written
   (4) \( f(x) = \cos^23x + \sin^22x - \cos^2x \)

The study of the validity of Bell's inequality reduces here to the study of the sign of \( f(x) \). \( f \)'s graph is shown in the figure for \( 0 \leq x \leq 90^\circ \).

We see that
   \( x = 0 \) \quad \rightarrow \text{Bell's inequality satisfied.} \\
   \( 0 < x < 30 \) \quad \rightarrow \text{Bell's inequality not satisfied.} \\
   \( 30 \leq x \leq 90 \) \quad \rightarrow \text{Bell's inequality satisfied.} 

Thus in QM, and in reality, Bell's inequality is satisfied by a continuum of directions \( a, b, c \); and it is falsified by a continuum of other directions.

7.4 Problem. The universality of Bell's inequality can only be the effect of an assumption. Somewhere in the proof in Section 4 we have explicitly or implicitly assumed universality. Where and how?

7.5 All the essential work in the proof of Bell's theorem is done in lemmas 4.9 and 4.11.

Inequality 4.9 is almost Bell's inequality, and it is universally quantified. Like Bell's inequality it is valid
for all directions \(a, b, c\). But Lemma 4.9 is based only on the assumption that with each photon we may associate a definite polarization value for all three directions \(a, b, c\) at the same time. This, in turn, is a consequence of Assumption H. Thus the universality is implicit in the assumption of hidden variables.

Inequality 4.11 is based on the locality assumption L. The role of locality in this connection may be described as follows. The locality assumption does not add to the universality; but it prevents that we loose the locality already won in Lemma 4.9.

We conclude that the universality is implicit in the assumption of hidden variables only.

7.6 To see more precisely how the universality arises out of Assumption H consider three directions \(a, b, c\) for which Bell's inequality fails. Let, e.g., the angles between \(a, b, c\) be as follows (cf. §6.3):

\[
\begin{align*}
V_{ab} &= 60^\circ, \\
V_{ab} &= 20^\circ, \\
V_{bc} &= 40^\circ.
\end{align*}
\]

We imagine a typical experiment with these directions. ("typical" here means that we assume that we are not victims of a statistical fluke.)

The first few lines of a record of the experiment may look as follows. (We eliminate all lines where the two instruments A and B are set in the same direction.)

<table>
<thead>
<tr>
<th>Setting</th>
<th>(P_A)</th>
<th>(P_B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ab)</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>(bc)</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>(ab)</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(ac)</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>(bc)</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>(ac)</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

In each line the two entries for the unmeasured direction are left open. These values can be chosen in two different ways for each line. If there are \(N\) pairs of photons in the experiment, there are \(2^N\) ways to complete the list. But no matter which of the \(2^N\) ways we choose, we obtain two 3-column lists. For any such list we can prove Bell's inequality just as in Section 4.

On the other hand, if we don't add values for the third direction, it is impossible to derive Bell's inequality for the given directions \(a, b, c\). For the data in the list are all from an experiment which falsify Bell's inequality.

Thus it is the very assumption that there is a value (no matter which) for the third direction which gives rise to Bell's inequality even for choices of directions for which it is false.

7.7 From the discussion in §§ 7.5-7.6 we see that we have the following two alternatives.

Alternative 1: It is incompatible with QM to assign a definite polarization or spin value to an elementary particle in 3 distinct directions. It is compatible with QM to assume definite polarization or spin values in 2 different directions.

Alternative 2: Bell's inequality (or rather Inequality 4.9) holds for the particles between S and the moment of the first measurement. The measurement of the particle, which arrives first to its measurement instrument, causes an immediate and non-local change in the polarization or spin of the twin particle such that the values now are in agreement with the QM inequality.

7.8 Remarks. (i) Alternative 7.7.1 is a key idea in my attempt to solve the problem raised by Bell's theorem.

(ii) So far it is still an open question whether Alternative 7.7.2 is tenable.
8. NON-LOCAL HIDDEN VARIABLES

8.1 In sections 4 and 6 it was shown that hidden variables and locality together are incompatible with QM and with experimental results. We may now ask if hidden variables and non-locality is a better alternative. This problem will be treated in the present section. We consider an experimental arrangement as in § 4.2.

8.2 Assumptions. We make the following assumptions.
Assumption H: We assume the existence of hidden variables as in Section 4.
Assumption -L: We assume the negation of the locality assumption. More precisely we assume that the setting of, or measurement with, one instrument can instantaneously cause a change in the value of a polarization variable of a distant photon.

8.3 Example. The idea of non-local hidden variable theories in connection with Bell's theorem can best be illustrated by an example. We use an experimental arrangement as in § 4.2 and measure the polarization variables of photon pairs; but we assume that instrument A is so much closer to S than instrument B is, that PA is measured before B is set.

The statistical predictions of QM are independent of the lengths SA and SB. Choose a, b, c so that Bell's inequality is violated, e.g.,
\[ V_{ac} = 20^\circ; \quad V_{bc} = 40^\circ; \quad V_{ab} = 60^\circ. \]
When PA and PB leave S, they have by Assumption H a certain polarization value for a, b, c. E.g.,

<table>
<thead>
<tr>
<th>Setting</th>
<th>PA</th>
<th>PB</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AB)</td>
<td>a b c</td>
<td>a b c</td>
</tr>
<tr>
<td>(bc)</td>
<td>+ + +</td>
<td>- + -</td>
</tr>
</tbody>
</table>

If the instruments A and B just read off the values which the polarization variables have when they leave S, then Bell's inequality follows inevitably.

To avoid Bell's inequality some change must take place with a probability > 0 in the value of at least one variable. The changes must occur with probabilities such that the measured results correspond to the predictions of QM. Since PA is measured before instrument B is set, the changes must occur in PB. In the given example, e.g., c in PB may change.

<table>
<thead>
<tr>
<th>Setting</th>
<th>PA</th>
<th>PB</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AB)</td>
<td>a b c</td>
<td>a b c</td>
</tr>
<tr>
<td>(bc)</td>
<td>+ + +</td>
<td>- + +</td>
</tr>
</tbody>
</table>

Or a may change but not c.

<table>
<thead>
<tr>
<th>Setting</th>
<th>PA</th>
<th>PB</th>
</tr>
</thead>
<tbody>
<tr>
<td>(AB)</td>
<td>a b c</td>
<td>a b c</td>
</tr>
<tr>
<td>(bc)</td>
<td>+ + +</td>
<td>+ + +</td>
</tr>
</tbody>
</table>

It is also possible that both change or none does. The b-value in PB, however, must not change to preserve the negative correlation.

We see that in order to have the right probabilities for the changes in PB, information about the setting of A at the moment of measurement is needed. Because of the spacelike separation this must be transmitted superluminally. Therefore we must allow for non-locality.
8.4 An example of a non-local hidden variable theory is the quantum potential approach to QM by Bohm and others. It is claimed to be able to duplicate all the predictions of standard QM.

With every particle is associated a wave. The relation between wave and particle is one of duality, not complementarity as in the, in many ways opposite, Copenhagen interpretation. This means that the wave and the particle are two distinct but closely related systems. The wave has the function of a pilot wave. It guides the movements and states of the associated particle. Every part of the wave is in instantaneous and non-local contact with the particle. This is needed to account for, e.g., the outcome of the Bell experiments. All the energy of the combined system of wave and particle is concentrated in the particle. The wave carries no energy at all. The wave acts by and large as a classical wave. The particle acts by and large as a classical particle.

8.5 Example. As a first example of the theory outlined in § 8.4 we consider a two-slit experiment. Electrons are generated one and one in the source S.

They are send towards the diaphragm D. The particle p acts like a classical particle and passes through exactly one of the slits T₁ and T₂, say T₁. The pilot wave also acts classically. It splits in two, one half wave following p through T₁ while the other half wave passes through T₂. When the two half waves arrives at the screen E, they interfere with each other. The resulting wave guides p towards the points where the amplitude is largest. The theory predicts correctly the characteristic interference pattern. (I have used electrons rather than photons in the thought experiment, because the quantum potential theory of the photon is more complicated than the theory of the electron.)

8.6 Example. In this example the theory of § 8.4 is applied to an experiment of Bell type.

Two protons are generated in the singlet state in S. As they interact with each other in S, their pilot waves are entangled. One part of p₁’s pilot wave follows p₁ while the other part follows p₂, and symmetrically for p₂’s pilot wave. When spin meter A is set to direction b and p₁ is measured in that direction, p₂’s pilot wave around p₁ "feels" this and guides instantaneously and non-locally the spin of p₂ towards values which will make the measurement results match the predictions of QM, as suggested in Example 8.3.

8.7 Remarks. A few critical remarks are made on the quantum potential theory.
(i) Several features of the theory appear to be ad hoc. E.g., why and how are a wave and a particle associated with each other? It remains a mystery.
(ii) A pilot wave carries no energy. This is necessary to avoid conflict with SR. In spite of that it can move and change the state of a massive particle. This is parapsychology, not physics.
(iii) The non-locality of the theory, as in Example 8.6, is incompatible with SR. This is proved in Theorem 8.9 below.
(iv) Two particles, which have interacted, form one system as in Example 8.6. The parts of the combined system are in immediate and non-local contact with each other. This idea of holism is impossible without non-locality. But by SR no non-locality can have observable consequences. Therefore the idea of holism in the quantum potential theory is untenable.

8.8 The critical remarks in §8.7 apply to the quantum potential theory. A priori we may imagine other non-local hidden variable theories which fare better. The following theorem shows, however, that no non-local hidden variable theory can solve the problem raised by Bell’s theorem.

8.9 Theorem. Any non-local hidden variable theory, which duplicates the predictions of QM, is incompatible with SR.

Proof:
Suppose there is such a non-local hidden variable theory T. Then T satisfies conditions H and \(-L\); and according to T there are directions a, b, c for which Bell’s inequality is false. We want to prove that \(T + SR\) is inconsistent.

We consider an experimental set-up as in Example 8.3. Choose the directions a, b, c such that Bell’s inequality is falsified. We assume, as in Example 8.3, that

(1) instruments A and B are set only after \(p_A\) and \(p_B\) have left \(S\);

(2) the distance \(SA\) is so much shorter than \(SB\) that the measurement of \(p_A\) is concluded before instrument B is set (relative to the laboratory frame).

As a consequence of (1) and (2) there are space-time regions \(R_A\) and \(R_B\) such that

(3) \(R_A\) contains the setting of A and the measurement of \(p_A\), and the same relation holds between \(R_B\), B, and \(p_B\);

(4) every event in \(R_A\) is spacelike separated from every event in \(R_B\);

(5) every event in \(R_A\) precedes every event in \(R_B\) relative to the laboratory frame.

In the experiment we use two inertial observers \(O_1\) and \(O_2\) to read off the instruments. \(O_2\) is at rest relative to the laboratory frame \(Ox_1t_1\). By SR and (4) there is an inertial frame \(Ox_2t_2\) such that relative to \(Ox_2t_2\) every event in \(R_B\) precedes every event in \(R_A\). Let \(O_2\) be at rest in \(Ox_2t_2\).

From these choices of frames we get the following consequences.

(6) In \(O_1\)'s frame of reference the measurement of \(p_A\) is completed before instrument B is set.

(7) In \(O_2\)'s frame of reference the measurement of \(p_B\) is completed before instrument A is set.

We let \(O_1\) observe the reactions of instrument A and \(O_2\) observe the reactions of instrument B. For then, by (6), the polarization value of \(p_A\) will be measured as it is, undisturbed by the setting of instrument B and the measurement performed on \(p_B\). Similarly, by (7), the polarization of \(p_B\) is measured as it is, undisturbed by the setting of A and the measurement performed on \(p_A\). The non-local corrections and correlations of the values of the polarization variables needed for a hidden variable theory to agree with QM are impossible.
A more direct derivation of Bell's inequality from Assumption H and SR is possible.

I. For any photon and any moment \( t \) the value of each of the polarization variables \( a, b, c \) at \( t \) is invariant. We may identify the polarization value at \( t \) with the ability to elicit a certain instrument reaction at \( t \). Since the deflection of an instrument (+1 or -1) at \( t \) is invariant, so is the polarization value for the given direction.

II. All of the polarization variables \( a, b, c \) in \( P_A \) (in \( P_B \)) have a constant value from \( P_A \) (\( P_B \)) leaves \( S \) until \( P_A \) (\( P_B \)) is measured. (This is 4.5(ii).)

Choose an arbitrary point \( Q_B \) between \( S \) and \( B \). Measure the polarization of \( P_A \) in, e.g., direction \( a \). Let the result be \( s \) (\( s = +1 \)). By SR there is an inertial frame \( F \) such that in \( F \) \( P_B \) is in the point \( Q_B \) after \( P_A \) is measured. By \( QM \) and experiments we know that \( P_B \) has polarization value -\( s \) in direction \( a \) when it is in \( Q_B \). By (I) this value is -\( s \) also in the laboratory frame. Since \( Q_B \) was arbitrary, \( P_B \) has the value -\( s \) in all points between \( S \) and \( B \). Therefore this value is constant. By a symmetrical argument we get that even \( P_A \)'s polarization values are constant.

III. For any of the directions \( a, b, c \), \( P_A \) and \( P_B \) have negatively correlated polarization values in the given direction.

Choose without loss of generality one direction, e.g., \( a \). Measure \( P_A \) and \( P_B \) in direction \( a \). By \( QM \) we get opposite results. By (II), \( P_A \) and \( P_B \) have had their values since they left \( S \).

To prove Lemma 4.9 we needed only Assumption H. Lemma 4.9 is therefore provable in a non-local hidden variable theory. To prove Lemma 4.11 we needed only perfect negative correlation (our Statement III) and Corollary 4.5(ii) (which is our Statement II). We now get Bell's inequality for the chosen directions \( a, b, c \) as a corollary to lemmas 4.9 and 4.11 just as in Section 4. But this contradicts the choice of the directions \( a, b, c \).

8.10 Corollary. Assume the existence of hidden variables and SR. Then Bell's inequality follows.

Proof: As a part of the proof of Theorem 8.9 we showed that Assumption H together with SR and perfect opposite correlation implies Bell's inequality. Since the negative correlation may be considered a quantum fact, the corollary follows.

8.11 Remark. Even the generalized Bell inequality of Section 5 could have been used in Theorem 8.9. The non-locality hypothesis here takes the following form: When \( P_A \) is measured, the hidden parameter \( \lambda \) changes non-locally to a new value \( \lambda' \). Thus in measuring \( P_A \) we observe the effect of \( \lambda \), while measuring \( P_B \) we observe the effect of \( \lambda' \).

To show this untenable we have two observers \( O_1 \) and \( O_2 \) as in § 8.9. \( O_1 \) observes \( P_A \) and measures the effect of \( \lambda \). \( O_2 \) observes \( P_B \) and measures likewise, in his frame of reference, the effect of \( \lambda \) (undisturbed by the other, still not performed, measurement).

8.12 We make a logical analysis of the argument. (For the meaning of the abbreviations see § 6.7.)

(1) \( H \land \neg L \rightarrow \neg SR \quad \text{Theorem 8.9} \)

I see no reason to question the validity of SR. Therefore,

(2) \( SR \)

(3) \( \neg (H \land \neg L) \quad (1), (2), \text{logic} \)

(4) \( H \land L \rightarrow B \quad \text{Bell's theorem} \)

(5) \( \neg B \quad \text{QM, experiments} \)

(6) \( \neg (H \land L) \quad (4), (5), \text{logic} \)

(7) \( \neg H \quad (3), (6), \text{logic} \)

Thus we may conclude that the assumption of hidden variables is false.

8.13 An alternative way to reach the same conclusion as in § 8.12 is to use Corollary 8.10 instead of Theorem 8.9.

(1) \( H \land SR \rightarrow B \quad \text{Corollary 8.10} \)

(2) \( \neg B \quad \text{QM, experiments} \)

(3) \( \neg (H \land SR) \quad (1), (2), \text{logic} \)
Corollary 8.10 may be written equivalently as
\[ \text{SR} \land \neg \text{H} \rightarrow \neg \text{H}. \]
Thus the falsehood of the hypothesis of hidden variables follows from unquestionable physical principles and results. An apparently ontological problem has been decided on the basis of purely physical principles.

8.14 In Section 7 we found two alternatives. Alternative 7.7.2 is just the idea that a non-local hidden variable theory may be compatible with QM. The results of the present section show that Alternative 7.7.2 is untenable. Therefore Alternative 7.7.1 is the only possibility. We will analyze and explore this possibility further in the remaining sections of this essay.

8.15 Bell (1964), d'Espagnat (1984), Herbert (1985), Rae (1986), Redhead (1987) and many others conclude from their analyses of Bell's theorem that the locality assumption is false. The results of the present section show that this inference is ill-founded. The issue of non-locality is irrelevant in connection with the problem raised by Bell's theorem.

8.16 We have derived \(-\text{H}\) from Bell's theorem and Theorem 8.9. Since I consider these two theorems correct, I also accept the conclusion \(-\text{H}\) wholeheartedly.

In § 6.7 we derived \(-\text{L}\) from Bell's theorem and the EPR argument. Since the EPR argument is felt to be less solid than the two theorems, the conclusion \(-\text{L}\) is vulnerable.

9 THE COPENHAGEN INTERPRETATION

9.1 There are at least three reasons for considering the Copenhagen interpretation (CI) of QM.
(i) We have derived \(-\text{H}\) and \(-\text{L}\). CI denies the existence of hidden variables; and because of the EPR argument Bohr was forced to accept a form of non-locality.
(ii) On the other hand we found the conclusion \(-\text{L}\) dubious. EPR was found to be the possibly weak link in the derivation of \(-\text{L}\). The EPR argument was an attack on CI. And Bohr developed CI further in response to EPR. CI may therefore contain ideas which can be useful in disarming EPR.
(iii) CI is the most influential of all interpretations of QM. That is one reason to check what it can do in connection with the Bell problem and whether it deserves its reputation.

9.2 The main originator of CI is Niels Bohr. His writings are characterized by intuitive power and the careful analysis of many concrete examples. In spite of this they are also characterized by a vagueness and ambiguity which makes them difficult to understand (at least for my small head). It contributes to the difficulties that the notion of complementarity sometimes plays a role in Bohr's thinking similar to the role played by dialectics in Marxist thinking: to camouflage muddle-headedness.

In spite of these difficulties I will attempt an interpretation of the Copenhagen interpretation.

9.3 Bohr's basic thesis is that the wave function \(\Psi\) of a quantum system \(p\) characterizes completely the physical state of \(p\). In this sense QM is complete. There is no deep reality under what can be computed in QM and measured. Thus, e.g., there are no hidden variables. QM is not in need of revision.
9.4 To state the fundamental principles of CI we need a distinction. **Static properties** of an elementary particle are properties which cannot vary with time, e.g., mass, electric charge, spin size. **Dynamic properties** are properties which can vary over time, e.g., position, velocity, spin direction.

9.5 CI is based on the following two fundamental principles:

**Principle A:** The working of measuring instruments must be accounted for in purely classical terms.

**Principle B:** Quantum systems cannot be thought of as having dynamic properties independently of the experimental arrangement.

9.6 The justification for Principle A given by Bohr (1958) is that "by the word experiment we can only mean a procedure regarding which we are able to communicate to others what we have done and what we have learnt". This implies that the description of the experimental arrangement and the experimental results must be given "in plain language, suitably refined by the usual physical terminology". According to Bohr this language is the language of classical physics.

9.7 To justify Principle B we consider as an example the delayed choice experiment. A pulse wave so short that it only contains one photon is generated in $S$. In the point $A$ a beam splitter $BS1$ (a half-silvered mirror) divides the wave packet in two halves, one following path $x$ and the other path $y$. The two paths meet in point $B$. In $B$ there is a second beam splitter $BS2$ which is mobile. It may be inserted as indicated in the figure or removed. In the two paths after $B$ there are photon multipliers $PMx$ and $PMy$.

Suppose $BS2$ is removed. Then the photons will be registered sometimes in $PMx$ and sometimes in $PMy$, randomly and evenly distributed between them. This is easily understood if we think of the photon as a particle which at $A$ with probability 0.5 follows path $x$ and not path $y$, and with probability 0.5 follows path $y$ and not path $x$. It is unintelligible if the photon behaves like a wave and splits in two half waves at $A$, one following path $x$ and the other path $y$.

Suppose now that $BS2$ is in place. Then it is possible to adjust the lengths of path $x$ and path $y$ such that all photons are registered in $PMy$ and none in $PMx$. This is unintelligible if the photons behave particlelike and follow only one path. It is easy to explain if the photons behave wave-like and split at $A$, one half wave following path $x$ and the other half path $y$. For then, by the occurrence of evanescent waves at $B$, we get destructive interference on the segment from $B$ to $PMx$ and constructive interference on the segment from $B$ to $PMy$.

The phenomena just described occur even if $BS2$ is inserted or removed only after the photon has passed $BS1$ and must have chosen whether to follow only one path, or split and follow both paths.

But how can each photon both follow exactly one of the two paths and split and follow both paths?
Bohr's answer is that the photon in itself can neither be ascribed wave nor particle properties. It is the photon + one experimental arrangement (BS2 in place) which can be ascribed the wave property. It is the photon + another experimental arrangement (BS2 removed) which can be ascribed the particle property. Since the two experimental arrangements logically exclude each other, no contradiction can arise.

The whole, consisting of quantum system + experimental arrangement, is called a phenomenon by Bohr. Strictly speaking it is phenomena only, which may be ascribed quantum properties. Quantum systems, like, e.g., elementary particles, are abstractions from phenomena. Thus quantum properties really are relations between quantum systems and measurement instruments. This relational conception of quantum properties is a cornerstone of CI.

9.8 The Principle of Complementarity. Assume we perform the abstraction process mentioned in § 9.7 and talk about quantum system in themselves. Then we can also ascribe properties like position and momentum or wave and particle properties to the systems in themselves. A pair of properties which are associated with logically incompatible experimental arrangements are called complementary.

Complementary properties are subject to strong restrictions. No elementary particle can be ascribed two complementary properties at the same time.

9.9 Examples. (i) As shown in § 9.7 wave and particle properties of an elementary system are complementary.
(ii) By the indeterminacy relations the position and momentum of a particle are complementary.
(iii) By the indeterminacy relations the spin values of a particle in two not collinear directions are complementary.
(iv) By the indeterminacy relations the polarization values of a photon in two not perpendicular directions are complementary.

9.10 Example. (i) A sequence of photons are prepared such that they are polarized at an angle of 45° to the horizontal and send through an HV-polariser.

By QM each photon will come out through the H-channel with probability 0.5, and through the V-channel with probability 0.5. According to CI a photon has a definite polarization value before it reaches the HV-polariser only in the 45°-direction. By interaction with the HV-polariser it assumes randomly a certain value (+1 or -1) in the H-direction.

(ii) We now modify the experimental arrangement as shown in the figure.

Now all photons passing through the H-channel of the first HV-polariser will be registered in the H-channel even by the second HV-polariser. Therefore each such photon has a definite polarization value H when it leaves the first HV-polariser. (At the same time it looses its sharp polarization value in the 45° direction.) Analogous conclusions hold for photons passing through the V-channel of the first HV-polariser.
A variable can be assigned a value before measurement iff the result of the measurement of the variable can be predicted with probability 1.

9.11 Example. As a last example we consider Bohr's answer to EPR. The EPR argument may be summarized as follows.

Let photon $p_A$ reach instrument $A$ before $p_B$ reaches $B$. Measure $p_A$ in direction $a$. Let, e.g., the result be +1. Then we can by QM predict with probability 1 that when $p_B$ is measured in direction $a$, the result will be -1. But by CI, $p_A$ has no definite polarization value in direction $a$ until the moment where it is measured. The same holds for $p_B$.

(EPR 1) But we have just, using QM, found a way to show that $p_B$ has a value in direction $a$ before it is measured or otherwise interfered with. This contradicts CI.

(EPR 2) By the locality assumption $p_B$ must have had this value before $p_A$ was measured. Therefore $p_B$ has in itself a sharp polarization value in direction $a$. Since $a$ was an arbitrarily chosen direction, this holds for any direction. This again contradicts CI.

(EPR 3) Set $A$ to direction $a$ and $B$ to direction $b$. Then we can measure $p_B$ directly in direction $b$. Indirectly we can infer, from the result at $A$ and the negative correlation predicted by QM, what measurement result we should have got had we measured $p_B$ in direction $a$ instead of direction $b$. Thus $p_B$ can be assigned a value in two non-perpendicular directions at the same time. This is in contradiction with CI.

Bohr's answer to EPR is based on two ideas.

Holism: When two particles have interacted (like the photons $p_A$ and $p_B$, or two protons in the singlet state), they must be considered one quantum system, no matter how far they are from each other, until they are separated by a measurement or other disturbance.

Wholeness: Quantum properties are relational. They are properties of quantum system + measurement instrument.

Using these ideas Bohr counters the EPR argument. (B 1) $p_A$ and $p_B$ have no sharp polarization values in direction $a$ when they leave $S$. When $p_A$ is measured it assumes randomly in interaction with instrument $A$ one of the values +1 and -1, say +1. Since $p_A$ and $p_B$ form one system, $p_B$ by the same interaction with instrument $A$ assumes, in accordance with the common wave function $\psi$ for $p_A$ and $p_B$, instantly the value -1. This instant correlation-at-a-distance is a form of non-locality forced upon CI by the EPR argument.

(B 2) The answer to (EPR 1) immediately disarms even (EPR 2).

(B 3) When $p_B$ has been measured in direction $a$, then by the principle of wholeness we can indeed assign a definite value for direction $a$ to the phenomenon ($p_B$ + instrument $B$ set to $a$). But we cannot assign a definite value for direction $a$ to the phenomenon ($p_B$ + instrument $B$ set to $b$).

9.12 If the above exposition of CI is confused, it only reflects the confusion in which my attempts to understand CI have left me. In §§ 9.13-9.16 I make a few brief critical remarks on CI.

9.13 It appears to be a too radical and unnatural measure to take to forbid the assignment of quantum properties to elementary particles and instead consider all such properties relational. From Example 9.7 it is clear that the apparent behaviour of each photon depends on the experimental arrangement. But such dependencies can be incorporated in dispositional terms.
Consider first a non-quantum example. Let S be a lump of sugar and T a piece of wood. Then we may say
S is dissoluble in water.
T is not dissoluble in water.
Solubility in water is normally treated as a dispositional property which S and T may or may not have. It is not
treated as a relation between S (or T) and water.

Dispositional terms can be analyzed by means of conditionals:
X is dissoluble in water
⇔ if X is placed in water, then X will dissolve.
I see no reason for not analyzing quantum properties as
dispositions in the same way. Thus
Proton p has spin a+ at t
⇔ p has the disposition at t to elicit the instrument re-
action a+ at t.
⇔ if p is tested for spin in direction a at t, then the
instrument reaction is +1.

In Example 9.7 each photon is disposed to elicit one type
of instrument reaction if BS2 is in place. It is disposed to
elicit another type of instrument reaction if BS2 is re-
moved. Because the antecedents are incompatible, no con-
tradiction will arise.

9.14 The idea of holism introduced in § 9.11 is completely
unwarranted and ad hoc.

Suppose a pair of protons PA and PB have been brought in
the singlet state by being forced into the same cell C in
phase space. Then, as long as PA and PB are in C, they are,
by Heisenberg's indeterminacy relations, even in principle
indiscernible. In that state they form physically one sys-
tem. But as soon as they are no longer in the same cell in
the phase space, there is no longer any physical justifica-
tion for considering them one system. They are two dis-
cernible and separated quantum systems which can interact
only inside the light cone.

9.15 Bohr's attitude that we must acceptQM as it is and
that QM is not in need of further foundations appears to me
to be unacceptable and sterile. QM is in many ways mysteri-
onous and counter-intuitive. The only way out of this situa-
tion is to develop the foundations of QM. Nothing precludes
that there are less naïve approaches to the foundations of
QM than attempts based on hidden variables.

9.16 The non-locality, in the form of instant correlations
at a distance, claimed by Bohr, is highly dubious. Such cor-
relations should be impossible without transfer of informa-
tion. Information can be identified with energy. But by SR,
energy cannot be transferred faster than light.

In the next example I examine more closely the relation-
ship between CI and SR.

9.17 Example. We consider an experimental arrangement as in
Theorem 8.9 with one change:

Instruments A and B are both set to measure in direction a
from an initial setting j. Thus A is set to a and PA mea-
sured before B is set, relative to the laboratory frame.
Again we have two initial observers O1 and O2 such that
(1) in O1's frame of reference the measurement of PA is com-
pleted before B is set;
(2) in O2's frame of reference the measurement of PB is com-
pleted before A is set.

Then by QM and CI:
In O1's frame: PB has a definite polarization value in di-
rection a immediately before it reaches B.
In $O_2$'s frame, $p_B$ has no definite polarization value in direction $a$ immediately before it reaches $B$.

The only way to avoid contradiction seems to be to make the physical state of $p_B$ non-invariant. Let $Q$ be a point between $S$ and $B$, sufficiently close to $B$. The coincidence of $p_B$'s position with $Q$ is invariant. In $O_1$'s frame of reference $p_B$ has a definite polarization value in direction $a$ when $p_B$ is in $Q$, say that the value is $a^+$. But this only means that the direction of $p_B$'s electric field is parallel with $a$. Such a parallelism should be invariant.

If we use pairs of protons in the singlet state, a similar conclusion is reached. The direction of rotation of an entity should be invariant.

There is some indication that what Bohr meant was that the definability of the physical state of a quantum system is non-invariant. We seem to have only two alternatives: (1) The physical state of $p_B$ in $Q$ is undefinable in $O_2$'s frame though $p_B$ actually is in a definite state in $Q$. (2) The physical state of $p_B$ in $Q$ is undefinable in $O_2$'s frame of reference and $p_B$ is not in a definite state in $Q$.

In Case (1), the indefinability is an expression of lack of knowledge. The probability ($P = 0.5$) in $O_2$'s frame that $+1$ will be the result of measuring $p_B$ in direction $a$ is subjective, contrary to what is claimed in CI. In Case (2) we are back with the problem mentioned above: The direction of the electric component of $p_B$'s electric field ought to be invariant.

9.18 The assumption of hidden variables $H$ of Section 4 is false according to CI. It is illegitimate to assign a definite polarization value to a photon in more than one direction at the same time. We will therefore get no three column list. It becomes impossible to derive Bell's inequality.

But CI has unacceptable consequences and limitations. We must find another no hidden variables theory.

9.19 Such a theory should satisfy the following conditions:
10 RECONSTRUCTION OF THE EPR ARGUMENT

10.1 There are at least two reasons for reconsidering the EPR argument.
(i) In § 6.7 we showed that
Bell's theorem + EPR & non-locality.
In Section 8 it was shown why non-locality is unacceptable,
at least in connection with hidden variables. It is more
probable that the faulty step is part of the EPR argument
than of the proof of Bell's theorem.
(ii) In the EPR argument, as it is exposed in Section 3, it
was concluded that locality implies hidden variables
\[ L \rightarrow H. \]
L can hardly be questioned; but H was proved to be false in
§§ 8.12-8.13. Therefore the EPR argument, in spite of its
intuitive appeal, must be partly wrong.

10.2 We assume locality.
Assumption L: Two systems (including quantum systems) cannot
interact with each other outside the light cone.

10.3 Experimental arrangement.

\[ \begin{array}{c}
\text{A} \\
\text{PA} \\
\text{S} \\
\text{PB} \\
\text{B} \\
\end{array} \]

We consider the same set-up as several times before. Pairs
of protons \( PA \) and \( PB \) are generated in the singlet state in
\( S \). Spin meters A and B are set to measure the spin of \( PA \)
resp. \( PB \) in an arbitrary direction \( X \). We assume that there
are spacelike separated spacetime regions \( R_A \) and \( R_B \) such
that \( R_A \) contains a segment of \( PA \)'s world line, the setting
of A and the measurement of \( PA \); and similarly \( R_B \) contains a
segment of \( PB \)'s world line, the setting of B and the measure-
ment of \( PB \) (see § 8.9).

The use of pairs of protons and spin is not essential. We
might have used other particles instead, or pairs of photons
and polarization.

10.4 We regard protons as individuals in their own right.
Spin is regarded as a dispositional property of a proton.
Spin is the disposition to elicit a certain instrument reaction.

Let us use the following symbols:
\[ X_t(p): \text{Proton } p \text{ is measured for spin in direction } X \text{ at time } t. \]
\[ X_t^+(p): \text{The result of measuring } p \text{ in direction } X \text{ at } t \text{ is } +1. \]
\[ X_t^-(p): \text{The result of measuring } p \text{ in direction } X \text{ at } t \text{ is } -1. \]
Then we can analyze the spin attribute as follows.
\[ p \text{ has spin } +1 \text{ in direction } X \text{ at } t \iff 
(X_t(p) \Rightarrow X_t^+(p)). \]

\[ \Rightarrow \] expresses an intuitive conditional, not material
implication. With material implication we should have
\[ -X_t(p) \Rightarrow (X_t(p) \Rightarrow X_t^+(p)) \]
\[ -X_t(p) \Rightarrow (X_t(p) \Rightarrow X_t^-(p)) \]
That is, if \( p \) is not measured for spin in direction \( X \) at \( t \),
then, at \( t \), \( p \) has both spin +1 and spin -1 in direction \( X \).
We do not want that.
\[ \Rightarrow \] is supposed to express a functional connection between
the two macrophysical events described by \( X_t(p) \) and
\( X_t^+(p) \). This connection is based on an intrinsic property,
the spin, of the proton \( p \).

Thus sentences like
\[ X_t(p) \Rightarrow X_t^+(p) \]
\[ (X_t(p) \Rightarrow X_t^+(p)) \vee (X_t(p) \Rightarrow X_t^-(p)) \]
are existential sentences asserting the existence of an intrin-
sic property of \( p \).
When the reference to time is inessential, we may skip the time index and write, e.g.,

\[ X(p) \rightarrow X^+(p) \]

10.5 The following three formulas express well-established QM facts and will be used freely in the proofs.

1. \[ X(p) \rightarrow (X^+(p) \lor X^-(p)) \]

If we measure a proton's spin in any direction, then we get, and can only get, one of the results +1 and -1.

2. \[ (X_S(p_A) \land X_L(p_B)) \lor ((X_S^+(p_A) \land X_L^-(p_B)) \lor (X_S^-(p_A) \land X_L^+(p_B))) \]

or simpler

3. \[ (X(p_A) \land X(p_B)) \lor ((X^+(p_A) \land X^-(p_B)) \lor (X^-(p_A) \land X^+(p_B))) \]

Each of (2) and (3) expresses the fact of perfect negative correlation of the measurement results for the same direction \( X \) at \( A \) and at \( B \).

10.6 Lemma. Assume \( L \). Then for any \( X \),

\[ X_t(p_B) \rightarrow (\forall s \geq t) [ (X_S(p_A) \rightarrow X_S^+(p_A)) \lor (X_S(p_A) \rightarrow X_S^-(p_A)) ] \]

Proof:

Arrange the distances \( SA \) and \( SB \) such that, relative to the laboratory frame, \( p_B \) is measured as least as early as \( p_A \). We assume

1. \[ X_t(p_B) \]

By 10.5(1) we get

2. \[ X_t^+(p_B) \lor X_t^-(p_B) \]

By 10.5(2) we can predict that for any \( s \geq t \), if \( X_S(p_A) \), then the result will be the opposite of what was obtained at \( B \). Using this and (2) we get

3. \[ (X_S(p_A) \implies X_S^+(p_A)) \lor (X_S(p_A) \implies X_S^-(p_A)) \]

It remains to understand the nature of the implication. Since for all \( s \geq t \), we can predict the result of the measurement \( X_S(p_A) \) and since at \( s \), \( p_A \) is, by Assumption \( L \), causally isolated from \( p_B \), \( p_A \) must in itself be disposed to elicit the predicted measurement result. Hence (3) may be written as

\[ (X_S(p_A) \rightarrow X_S^+(p_A)) \lor (X_S(p_A) \rightarrow X_S^-(p_A)) \]

10.7 Lemma. Assume \( L \). Then for any \( X \),

\[ X(p_B) \rightarrow (\forall s \geq t) [ (X(p_A) \rightarrow X^+(p_A)) \lor (X(p_A) \rightarrow X^-(p_A)) ] \]

Proof:

Assume \( X_t(p_B) \). By Lemma 10.6,

1. \[ (X_L(p_A) \rightarrow X_L^+(p_A)) \lor (X_L(p_A) \rightarrow X_L^-(p_A)) \]

Consider any moment \( s \) such that

\[ 0 < s < t \]

where \( 0 \) is the moment when \( p_A \) and \( p_B \) are generated in \( S \). We may assume that \( p_A \) in the interval \( [s,t] \) is causally isolated from \( p_B \) and the working of instrument \( B \). As a consequence of this \( p_A \) must already at \( s \) have the spin value in direction \( X \) which is later revealed by instrument \( A \). For otherwise the opposite correlation of spin values obtained at \( A \) and \( B \) should not occur with probability 1. Thus

\[ X_t(p_B) \rightarrow (\forall s \geq t) [ (X_S(p_A) \rightarrow X_S^+(p_A)) \lor (X_S(p_A) \rightarrow X_S^-(p_A)) ] \]

or simpler

\[ X(p_B) \rightarrow (\forall s \geq t) [ (X(p_A) \rightarrow X^+(p_A)) \lor (X(p_A) \rightarrow X^-(p_A)) ] \]

10.8 Lemma. Assume \( L \). Then for any \( X \),

\[ X(p_A) \rightarrow (\forall s \geq t) [ (X(p_A) \rightarrow X^+(p_A)) \lor (X(p_A) \rightarrow X^-(p_A)) ] \]

Proof:

Assume \( X_t(p_A) \). Using \( L \) and the sure negative correlation of the measurement results at \( A \) and \( B \) we get, by a similar argument as in the proof of Lemma 10.7, that \( p_A \) has a constant spin value in direction \( X \) from it leaves \( S \) until the moment \( t \) of measurement.

10.9 Theorem. Assume \( L \). Then for any \( X \),

\[ (X(p_A) \lor X(p_B)) \rightarrow (\forall s \geq t) [ (X(p_A) \rightarrow X^+(p_A)) \lor (X(p_A) \rightarrow X^-(p_A)) ] \]

Proof:

This is an immediate corollary to lemmas 10.7 and 10.8.
10.10 **Realism** may be defined as follows. 
An interpretation of a theory $T$ is **realistic** iff, according to the interpretation, a correctly made measurement of any observable $v$ pertaining to an individual of theory $T$ reveals a value which $v$ has in itself independent of the measurement.

10.11 **Examples.** (i) An example of a realistic interpretation of QM is the quantum potential approach described in Section 8. Here all observables have at any time a sharp value. A correct measurement reveals precisely this value. (ii) The Copenhagen interpretation is a non-realistic interpretation of QM. In Example 9.10 the photon $p$ has a definite polarization in the $45^\circ$ direction only. When we measure $p$ for HV-polarization, the value we get (H+ or H-) is not a value which the variable has in itself independent of the measurement.

10.12 What Theorem 10.9 says is that for any direction $X$, in which we directly or indirectly measure the spin of $P_A$, it is true that $P_A$ has a definite spin value in direction $X$. By Lemma 10.8, a direct measurement reveals exactly that value. By 10.5(3) and Lemma 10.7, an indirect measurement reveals the same value.

Thus the only possible interpretation of Theorem 10.9 is that the locality assumption $L$ forces a realistic interpretation of QM upon us.

10.13 By 10.5(3) and Theorem 10.9 we may assign a definite spin value to a proton $p$ in two different directions: the one we measure directly and the one we measure indirectly. It is, however, impossible even to assume a sharp value for $p$ in a third direction at the same time. For otherwise we get a three column list; and Bell's theorem follows immediately as in Section 4.

We therefore have the following corollary.

10.14 **Corollary.** Let $a$ and $b$ be two independent directions in which we measure the spin of a proton $p$. Then it is impossible to assume that $p$ has a spin value in a third direction $c$ at the same time.

10.15 Consider a selection of in pairs independent directions $a$, $b$, $c$, .... For each direction we get two sentences expressing the two possible spin values in that direction.

<table>
<thead>
<tr>
<th>Direction</th>
<th>Spin +</th>
<th>Spin -</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>$a(p) \Rightarrow a+(p)$</td>
<td>$a(p) \Rightarrow a-(p)$</td>
</tr>
<tr>
<td>$b$</td>
<td>$b(p) \Rightarrow b+(p)$</td>
<td>$b(p) \Rightarrow b-(p)$</td>
</tr>
<tr>
<td>$c$</td>
<td>$c(p) \Rightarrow c+(p)$</td>
<td>$c(p) \Rightarrow c-(p)$</td>
</tr>
</tbody>
</table>

Suppose that the sentences obey classical logic. Then each of them must be assigned one of the truth values 0, 1. But by Corollary 10.14 we cannot have a 1 in more than two directions. The only possibility is to assign a 0 to both sentences in all directions except two. E.g.,

- $a(p_A) \Rightarrow a+(p_A)\quad a(p_A) \Rightarrow a-(p_A)$
- $b(p_A) \Rightarrow b+(p_A)\quad b(p_A) \Rightarrow b-(p_A)$
- $c(p_A) \Rightarrow c+(p_A)\quad c(p_A) \Rightarrow c-(p_A)$

But then we get, by 10.5(3), for the twin proton $P_B$

- $a(p_B) \Rightarrow a+(p_B)\quad a(p_B) \Rightarrow a-(p_B)$
- $b(p_B) \Rightarrow b+(p_B)\quad b(p_B) \Rightarrow b-(p_B)$
- $c(p_B) \Rightarrow c+(p_B)\quad c(p_B) \Rightarrow c-(p_B)$

This is incompatible with Corollary 10.14.

We may therefore conclude that sentences like $a(p) \Rightarrow a+(p)\quad a(p) \Rightarrow a-(p)$ do not obey classical logic. They cannot all be assigned truth values simultaneously.

10.16 **Corollary.** Assume $L$. For each direction $X$, we choose to consider,
Indeed, this disjunction holds for exactly two directions simultaneously.

Proof:
By Lemma 10.8, \( p_A \) has a definite spin value in direction \( X \) from \( p_A \) leaves \( S \) until the moment of measurement in direction \( X \). But the truth of the consequent in Lemma 10.8 depends only on the correspondence with the local reality in \( p_A \) and not on the future event \( X(p_A) \).

On the other hand, the disjunction can be thought of as holding for at most two directions simultaneously. For otherwise we get a set of possible three column lists; and Bell's inequality can be derived as in Section 4.

10.17 The spin of a proton can be considered and measured in each of a continuum of directions. But the real spin of the proton is only two-sided. We can measure it in only two directions; and we can think of it in only two directions simultaneously without contradiction.

The spin may be compared to a polyhedron. We can think of only two of its sides as real. We can register only two sides as real. All other sides do not belong to reality. We may choose ourselves (by setting the measurement instruments) which sides belong to reality and which not.

The basic problem in the foundations of QM is to find a model of the Universe and a notion of reality which explains this feature as well as other odd features of QM.

10.18 Objection. Cannot Lewis' (1973) theory of counterfactuals be used to show that Corollary 10.13 is false and that therefore the above sketched solution to the Bell problem is misguided?

Assume that \( p \) has been measured directly in direction \( a \) and indirectly in direction \( b \). To examine the truth value (in this world) of the counterfactual conditionals

\[
\begin{align*}
    c(p) &\Rightarrow c+(p) \\
    c(p) &\Rightarrow c-(p)
\end{align*}
\]

we consider the nearest \( c(p) \)-worlds. As shown in the proofs of lemmas 10.7–10.8, the spin value for a given direction is definite and constant from it leaves \( S \). Therefore we must have either \( c+(p) \) in all the nearest \( c(p) \)-worlds or \( c-(p) \) in all the nearest \( c(p) \)-worlds. Therefore

\[
(c(p) \Rightarrow c+(p)) \lor (c(p) \Rightarrow c-(p)).
\]

\( p \) has been shown to have a definite spin in three different directions simultaneously, and Bell's inequality follows.

10.19 Answer: Since \( p \) is measured in directions \( a \) and \( b \), \( p \) has a definite spin in these directions, while the spin variable \( S_c \) for direction \( c \) has no value. A variable without value is a random variable. Thus \( S_c \) is in the given \( a(p) \)- and \( b(p) \)-world a random variable. Therefore there is a nearest \( c(p) \)-world where we have \( c+(p) \), and there is a nearest \( c(p) \)-world in which we have \( c-(p) \). (In each of these \( c(p) \)-worlds, \( S_c \) is a deterministic variable by Theorem 10.9.)

Lewis' theory of counterfactuals cannot be used to show the ideas developed in this section inconsistent.

10.20 So far we have proved two of EPR's conclusions.
(i) We have in Theorem 10.9 derived the realism advocated by EPR.
(ii) We have shown that a proton \( p_A \) has a definite spin value in two different directions \( a \) and \( b \) simultaneously, immediately before \( p_A \) interacts with instrument \( A \). Instrument \( A \) set to direction \( a \) and instrument \( B \) set to direction \( b \), measure these values directly resp. indirectly. Thus \( p_A \)'s spin value in direction \( b \) is the result we should have obtained at instrument \( A \) if \( A \) had been set to direction \( b \) instead of direction \( a \).

10.21 In § 3.12 we gave an argument for hidden variables. It was based on the conclusion of realism or the EPR argument and the universalization principle. This principle says that the following is a valid mode of inference:

For each \( x \) \( P(x) \rightarrow \) For all \( x \) \( P(x) \).

Corollaries 10.14 and 10.16 show that the universalization principle is not valid in QM.

The structure of the argument of § 3.12 is
Realism + Universalization → Hidden variables or symbolically,
(1) \( R \land U \rightarrow H \).
(1) is correct, but \( U \) (and \( H \)) false.

10.22 We can now extend the earlier analysis in sentential logic of the argument. We use the following abbreviations:
\( H \): The assumption of hidden variables.
\( L \): The locality assumption.
\( B \): Bell's inequality.
\( SR \): The special theory of relativity.
\( R \): Realism in the sense of §§ 10.10, 10.12 and 10.16.
\( U \): The principle of universalization.

I have no doubts about \( SR \) and \( L \) and use them freely as premises in the deduction.

(1) \( L \rightarrow R \) Reconstructed EPR, § 10.16
(2) \( H \land L \rightarrow B \) Bell's theorem
(3) \( B \) QM, experiments
(4) \( \neg(H \land L) \) (2), (3), logic
(5) \( H \land \neg L \rightarrow SR \) Theorem 8.9
(6) \( SR \)
(7) \( \neg (H \land \neg L) \) (5), (6), logic
(8) \( \neg H \) (4), (7), logic
(9) \( R \land U \rightarrow H \) § 10.21
(10) \( \neg (R \land U) \) (8), (9), logic
(11) \( L \)
(12) \( R \) (1), (11), logic
(13) \( \neg U \) (10), (12), logic

Thus we may conclude
\( R \land L \land \neg U \land \neg H \).

10.23 A widespread interpretation of Bell's theorem is that the combination of locality and realism is incompatible with QM. The analysis of § 10.22 shows that this is erroneous. It is only if we add \( U \) also that we come into conflict with QM. The results of Section 7 show that there is no way to derive Bell's inequality from \( R \) and \( L \) alone. An assumption of universality is implicit in Bell's inequality and must be introduced in some form into the proof, explicitly or implicitly.

10.24 d'Espagnat (1984) purports to have given a proof by induction that every proton has a definite spin value in three directions \( a, b, c \) simultaneously.

We consider the usual experimental set-up with a large number \( N \) of pairs of protons \( (p_A, p_B) \) in the singlet state. First set instruments \( A \) and \( B \) to direction \( a \). Measure \( N/3 \) pairs for this setting. Since we always get negatively correlated spin values, we can infer that the \( 2N/3 \) protons have the obtained values before the moment of measurement. By induction we may infer that all the \( 2N \) protons have a definite spin value in direction \( a \). Similarly it is shown by induction that all the protons have a definite spin value in direction \( b \) and in direction \( c \). d'Espagnat infers that all the protons have a definite spin value in directions \( a, b, c \) simultaneously.

10.25 Answer. One can, indeed, show by induction that all protons have a definite spin value for each of the directions \( a, b, c \) separately. But for the last step of argument 10.24, to infer that the protons have a definite spin value for all directions \( a, b, c \) simultaneously, we need the principle of universalization. Because of the slippery and indeterminate character of quantum reality this principle is not valid.

d'Espagnat's often repeated claim that
Realism + Induction → Bell's inequality
is unwarranted. Without an explicit or implicit (and unjustified) universalization operation it is impossible to derive Bell's inequality.

10.26 We have shown that the assumption of universality, implicit in the assumption \( H \) of hidden variables of Section 4, is false; and therefore Assumption \( H \) itself is false. But
what is wrong in the proof of the generalized Bell inequality given in Section 5?

10.27 It is easily seen that even the generalized Bell inequality depends on an implicit universality assumption. Where in the proof is this assumption made?

Note first that the weak assumption (WH) of hidden variables is much weaker than Assumption H of Section 4. Assumption H implies that all observables have simultaneously definite and sharp values. The hidden variable \( \lambda \) of Section 5 is much weaker. It merely denotes the states (whatever they are) which together with the setting and working of the measurement apparatuses determine the probability of the possible measurement outcomes. We may, e.g., let \( \lambda \) denote the initial conditions in \( S \) during the generation of \( p_A \) and \( p_B \). It is difficult to deny that there are such initial conditions. Moreover, no assumption is made in the proof about the strength of \( \lambda \)'s influence on the probabilities of the outcomes. \( \lambda \) may even be so weak that it exerts no influence at all, such that, e.g.,

\[
P(A|\lambda,a,b,B) = P(A|a,b,B).
\]

It is still possible to derive Bell's inequality as in Section 5. It is therefore not possible to question the validity of Assumption WH.

But for inequality 5.7(7) to be defined, all of

\[
(1) \quad C(\lambda,a,b), \quad C(\lambda,a,b'), \quad C(\lambda,a',b), \quad C(\lambda,a',b')
\]

must be defined simultaneously.

According to 5.7(5),

\[
(2) \quad C(\lambda,a,b) =_{df} \sum_{A,B} A \cdot B \cdot P(A,B|\lambda,a,b)
\]

Therefore the correlation coefficients \( (1) \) are all defined iff for \( A,B = \emptyset \),

\[
P(A,B|\lambda,a,b)
\]

\[
P(A,B|\lambda,a',b)
\]

\[
P(A,B|\lambda,a,b')
\]

\[
P(A,B|\lambda,a',b')
\]

are all defined simultaneously.

Suppose this is the case. Consider the case where we have perfect negative correlation when instruments A and B are set to the same direction. By § 5.13 we have determinism. Therefore each of the 16 probabilities in (3) is equal to 0 or equal to 1. Now it is easy from the values of the probabilities in (3) to derive the spin values for \( p_A \) and \( p_B \) for all of the directions \( a, a', b, b' \). This is incompatible with corollaries 10.14 and 10.16.

Thus it is by assuming that all the correlation coefficients \( (1) \) are simultaneously definable that we make the universality assumption.

10.28 One conclusion of the EPR argument was that QM must be interpreted realistically. We have reached the same conclusion in this section. Another conclusion was that all observables have definite values simultaneously. We have shown that this conclusion is unjustified and false.

The main conclusion of the EPR argument was that QM is incomplete. Is this conclusion correct? (It is worth noting that the incompleteness of QM in EPR (1935) is a corollary to the conclusion of realism.)

10.29 To answer this question we compare with two concepts of completeness from mathematical logic.

A first-order theory \( T \) is formally complete iff for every sentence \( S \) of \( L(T) \), if \( T \not\vdash S \) then \( T + S \) is inconsistent.

\( T \) is deductively completed iff for every sentence \( S \), which is true in all the intended interpretations of \( T \), \( T \vdash S \).

10.30 Suppose there is an extension of QM in which it is possible to show that every pair \( (p_A, p_B) \) has definite spin values in some direction, e.g., direction \( a \), in accordance with EPR's intentions. Let \( S \) be a sentence which expresses that all \( p_A \) and \( p_B \) have definite spin values in direction \( a \). Choose directions \( b \) and \( c \) such that \( V_{ab}, V_{bc}, V_{ac} \) falsify Bell's inequality (see Section 6). Set instruments A and B to measure in direction \( b \) and \( c \), switching between them. With a sample of \( N \) proton pairs we may associate \( 2^N \) possible
three column lists compatible with the measurement results. Therefore in QM + S we can derive Bell's inequality for the directions a, b, c as in Section 4. But in QM alone we can derive the negation of Bell's inequality for directions a, b, c as in Section 6. Therefore QM + S is inconsistent.

We may therefore conclude that QM is not formally incomplete.

10.31 It is clear that the kind of incompleteness referred to by EPR is deductive incompleteness. The intended interpretation of QM is a fragment of physical reality. An element of this fragment of reality cannot be shown in QM to exist. QM must therefore be deductively incomplete.

It is, however, utterly doubtful how useful the concept of deductive incompleteness defined in § 10.29 is in connection with QM for the following reasons.

(i) In § 10.30 we saw that any attempt to repair the alleged incompleteness leads to a contradiction. It is hardly meaningful to talk about an incompleteness which cannot be improved.

(ii) The notion of deductive completeness of § 10.29 is adjusted to static models and realities. But, as shown in § 10.17, unmeasured quantum reality has a slippery and indeterminate character. This seems to demand other concepts of completeness.

(iii) As shown in § 10.15 and § 10.21, QM calls for a logic different from classical logic. No doubt this requires different metalogical concepts.

11 SPEECHIVE REMARKS

11.1 The "solution" most commonly proposed to the problem raised by Bell's theorem is to postulate "correlation at a distance". This is highly mysterious to say the least.

In the preceding sections I have tried to show that the solution is the following: Two conjugate variables may be assigned definite values at the same time. But if we have three variables which are conjugate in pairs, then they cannot all be assumed to have definite values simultaneously.

This is only a negative result. We have found a way of blocking the derivation of Bell's inequality. The paradoxical character of Bell's theorem stems from the apparent naturalness of its assumptions. What we want are positive insights into the nature of reality and the foundations of QM which make the results of QM look natural and an assumption (H) of Bell's theorem look unnatural. In this section, I sketch some vague, speculative ideas in this direction.

11.2 For any pair of conjugate variables there are indeterminacy relations. The best known are Heisenberg's relations for position and momentum and for time and energy

\[ \Delta x \Delta p_x \geq \hbar \]
\[ \Delta t \Delta E \geq \hbar \]

The spin components \( S_a \) and \( S_b \) of the spin of a particle in two non-collinear directions a and b are also conjugate and governed by an indeterminacy relation.

11.3 The two most common interpretations of the indeterminacy relations are the following.

(i) The indeterminacy relations are interpreted as statistical dispersion relations. Consider a large sample of identically prepared particles. On each particle measure two conjugate variables \( u \) and \( v \) immediately after each other. Then
the spread $\Delta u$ and $\Delta v$ in the measurement results obtained must satisfy the indeterminacy relation for $u$ and $v$.

Prepare, e.g., a sample of electrons which all have spin $a^\ast$. Let $b$ be orthogonal to $a$. Measure each electron for spin in direction $b$. Then the spin values $S_b = +1$ and $S_b = -1$ will be approximately evenly distributed among the electrons. Thus the dispersion of the $S_b$ values in this context is maximal.

(ii) The indeterminacy relations are interpreted as uncertainty relations. According to this interpretation, it is, even in principle, impossible to measure the simultaneous values of two conjugate variables pertaining to a quantum system with smaller margins of error than what the indeterminacy relations allow. This interpretation presupposes that the two conjugate variables have sharp, simultaneous values.

11.4 The dispersion interpretation is weaker and agreed on by almost everybody. It is, however, tempting to strengthen it to the uncertainty interpretation.

There is, however, an objection to the last-mentioned interpretation. By the EPR method described in sections 3 and 10 it is possible to measure the values of two conjugate variables exactly. Consider, e.g., the spin components of protons in two directions $a$ and $b$. Prepare a pair of protons ($P_a$, $P_b$) in the singlet state. Measure $P_a$ in direction $a$. Measure $P_b$ in direction $b$. Infer by the negative correlation that $P_a$ has the opposite spin value of $P_b$'s value in direction $b$. Now we have measured the exact simultaneous values of $P_a$'s spin components in direction $a$ and direction $b$ immediately before $P_a$ interacted with instrument $A$.

11.5 The solution proposed here is to distinguish between direct and indirect measurements. In the example of § 11.4 we measured $P_a$'s spin in direction $a$ directly. We measured $P_a$'s spin in direction $b$ indirectly by making a direct measurement on $P_b$.

If we use indirect measurement, it is possible to measure two conjugate variables exactly. If we only apply direct measurements, i.e., make measurements on only one particle, then the measurement results will be marred by the uncertainties given by the indeterminacy relations. The possibility of exact indirect measurements makes sure that the sharp values for two conjugate variables are defined simultaneously.

Indeterminacy relations are uncertainty relations concerning direct measurements.

11.6 Indeterminacy relations are thus impossibility results. They express the impossibility of determining by direct measurements alone the exact simultaneous values of two conjugate variables. Because of the possibility of indirect measurements it is, however, possible and consistent to assume that two conjugate variables have in themselves definite and sharp values.

11.7 According to the analysis in the preceding sections, Bell's theorem also implies an impossibility result. It implies the impossibility of even assuming that more than two of a set of variables conjugate in pairs have values simultaneously. All variables, except two, must be indeterminate. This kind of indeterminacy we may call theoretical indeterminacy, as opposed to the traditional experimental indeterminacy.

Thus Bell's inequality can be derived only if we violate a form of indeterminacy.

11.8 Problem. Bell's theorem arises, because the nature of the quantum mechanical indeterminacy is not well understood. We need a deeper understanding of this phenomenon. It ought to be possible to give a natural and unifying theory of experimental and theoretical indeterminacy.

Heisenberg's indeterminacy relations were shocking when they appeared. They are still not understood. The theoretical indeterminacy, claimed in § 11.7 to exist, is maybe even more shocking. Some variables can, under certain circumstances, not even be thought to have a value. I believe that
an understanding of reality is possible which makes both forms of indeterminacy natural and obvious.

11.9 Problem. In Section 10 we concluded that QM demands realism. But, as we saw, quantum reality has a special slippery and indeterminate character. We need a suitable concept of reality.

11.10 One possible idea about reality is the following: Reality is operational impossibility.

11.11 Examples. (i) A table is a real object. Its being real can be identified with a set of operational impossibilities. E.g., it is impossible to pierce the table-top with one's hand.

(ii) One can easily move one's hand through the air, though the air is also real. But one feel's the air resistance. The movement is impossible without expenditure of energy to overcome this air resistance.

(iii) The height of the table being 85 cm is part of reality. This piece of reality can be identified with the impossibility that a correct measurement operation should give any other result than 85 cm.

11.12 If we measure a proton \( p_A \)'s spin in direction \( a \), the result reflects a value which \( p_A \) has had since it left the source \( S \) as shown in Section 10. Therefore it is impossible that a correct measurement operation should give any other result. The spin component in direction \( a \) is part of reality.

11.13 Another operational impossibility occurs in connection with spin measurements (and other quantum measurements). By the indeterminacy relations it is impossible directly to measure the spin component in more than one direction. It is impossible directly or indirectly to measure more than two spin components. These operational impossibilities are also shaping parts of reality.

11.14 We get the following picture. The two variables we choose to measure are parts of reality. The impossibility imposed by the indeterminacy relations may be pictured as walls. The walls are themselves parts of reality. They enclose the two measured variables in reality but exclude all other variables from reality. Since the latter variables are not part of reality, they have no values.

The walls are mobile. We can ourselves decide where to place them, and thereby decide which two variables to enclose in reality and which to exclude. The form of reality is not decided before the moment of measurement, though the content is. This gives quantum reality its dynamic, slippery, and indeterminate character.

11.15 Another possible idea about reality is the following: Reality is the universe as seen from the inside.

11.16 It is an important fact that it is only possible to see a part of the universe from the inside. The spin of a proton, for instance, may be seen from any of continuum many sides. But it is only possible to see (and think of) the spin from two sides simultaneously.

11.17 Reality is smaller than the universe. It is only a segment. Any part of the universe may belong to reality. But due to the indeterminacy not all parts can belong to reality. Until the moment of measurement, or other macrophysical interaction, it is uncertain which parts of the universe are also parts of reality, and which not.

11.18 In classical physics we have reality = the universe.

In QM we have reality ⊂ the universe.

Reality is not a static subsystem of the universe. It is growing all the time. This gives quantum reality its dynamic, slippery, and indeterminate character.
11.19 When one wants to develop a theory, in this case the foundations of QM and quantum reality, models are useful. In §§ 11.20-11.22, I propose three models.

11.20 The situation in experiments concerned with the spin components of a pair of protons may be compared with a double kaleidoscope.

The cylinders $P_A$ and $P_B$ contain pictures $a$, $b$, $c$, ... Each picture shows either a '+' or a '-'. $P_A$ and $P_B$ can separately be rotated so that any of the pictures $a$, $b$, $c$, ... is in the panels $A$ resp. $B$. Thus the spectator can choose to watch anyone of the pictures on $P_A$ and anyone of the pictures on $P_B$. It is, however, only possible to watch one picture from each cylinder. For as soon as you watch one picture the other pictures on the cylinder are changed randomly. By a mechanism the pictures on the two cylinders are oppositely correlated, as indicated on the figure, before the spectator watch any of them.

Therefore two pictures on each cylinder may be considered parts of reality (here represented by the rectangle): the picture inspected directly, and the picture the content of which may be inferred indirectly. The other pictures are not parts of reality, i.e., inside the rectangle.

Of course, in at least one respect the model is misleading. The cylinders $P_A$ and $P_B$ are ordinary objects. No contradiction results from assuming that each of them has a picture in all of the squares $a$, $b$, $c$, ... simultaneously.

11.21 Another model for the spin experiment is shown in the figure.

$P_A$ and $P_B$ consists of three tenons $a$, $b$, $c$. In each tenon is engraved a '+' or a '-'. The signs are oppositely correlated. In the points $G$ and $H$ two pieces of cloth are fastened. One piece of cloth may be stretched and moved such that the sign of any one of the tenons $a$, $b$, $c$ at $P_A$ can be felt through the cloth. It is, however, not possible to stretch the cloth so much that two tenons can be felt simultaneously. Similarly, exactly one arbitrary tenon of $P_B$ can be inspected through the other cloth.

Two tenons at each place belong to reality: the one inspected directly and the one whose sign is inferred indirectly. The two pieces of cloth also belong to reality, corresponding to the restricting effects of the indeterminacy.
relations. The two pieces of cloth together exclude the third tenon from reality.

Again the model is misleading in at least one respect. Tenons are ordinary objects. No contradiction arises from assuming that even the third tenon has a definite sign.

11.22 Elementary particles are individuals but not objects. Bernays set theory is a theory which contains individuals that are not objects. Proper classes are individuals in Bernays set theory since they are quantified over. But they are not objects. Treating them as objects should imply considering them as sets. The traditional set paradoxes should follow immediately.

It may be hoped that an analysis of Bernays set theory will give ideas which are useful in connection with the foundations of QM.

11.23 Problem. In Section 10 we showed that sentences like
\[ X(p) \Rightarrow X+(p) \quad X(p) \Rightarrow X-(p) \]
do not obey classical logic. We need an elaboration and development of this new type of logic.

11.24 The most basic logical feature of these sentences is that among the continuum many pairs of sentences obtained from
\[ X(p) \Rightarrow X+(p) \quad X(p) \Rightarrow X-(p) \]
by letting \( X \) vary through the set of possible settings of the spin meter, only two pairs can have truth values. All other pairs of sentences are indeterminate.

This is just another way of saying that only two of the spin component variables can have values simultaneously. This, in turn, is, as sketched earlier in the present section, a consequence of the indeterminacy relations. It follows that the indeterminacy relations are parts of logic as much as they are of physics. There are no sharp boundaries between logic and physics.

11.25 Let us call this logic quantum conditional logic (QCL). QCL must have at least the following two features.

(i) QCL must have two universal quantifiers
For each \( x \) (separately)
For all \( x \) (simultaneously).

Classically we have
(1) For each \( x \) \( P(x) \) \( \iff \) For all \( x \) \( P(x) \)
In QCL we have, in contrast,
(2) For all \( x \) \( P(x) \) \( \rightarrow \) For each \( x \) \( P(x) \)
(3) For each \( x \) \( P(x) \) \( \not\rightarrow \) For all \( x \) \( P(x) \)
A counterexample verifying (3) is given in § 10.21.

(ii) Three sentences may be true separately without their conjunction having a truth value. E.g., it may be that
\[ a(p) \Rightarrow a+(p) \]
\[ b(p) \Rightarrow b-(p) \]
\[ c(p) \Rightarrow c-(p) \]
are each true taken separately. But the conjunction
\[ (a(p) \Rightarrow a+(p)) \land (b(p) \Rightarrow b-(p)) \land (c(p) \Rightarrow c-(p)) \]
has no truth value since the three conjunctions cannot all have a truth value simultaneously.

11.26 It is easy to show that collections of elementary particles cannot be Cantorian sets. To show this we consider the same experimental arrangement as in Section 4.

We make the following three assumptions.

Assumption S: Collections of photons are Cantorian sets.
Assumption L: Locality as in Section 4.
Assumption R: We assume realism in the sense of Corollary 10.16. For any direction \( X \) separately, the polarization component for a photon in direction \( X \) has a definite value.

Consider a sample of \( N \) pairs of photons \((P_A, P_B)\). Define
\[ M_{a^+} = \{P_B | a(P_B) \Rightarrow a+(P_B)\} \]
\[ M_{a^-} = \{P_B | a(P_B) \Rightarrow a-(P_B)\} \]
Analogously we define \( M_{b^+}, M_{b^-}, M_{c^+}, M_{c^-} \). By Assumption R these six sets are all well-defined.

Because of Assumption S we can apply the intersection operation unlimited, e.g.,
\( M_{a^+} \) is the set of all \( P_B \) having \( a^+ \).
\[ M_a^+ \cap M_b^- \] is the set of all \( p_B \) having \( a^+ \) and \( b^- \).
\[ M_a^+ \cap M_b^- \cap M_c^+ \] is the set of all \( p_B \) having \( a^+ \), \( b^- \) and \( c^+ \).

Let \( cd \) denote the cardinality function. Then, as is easily seen,

\[
\begin{align*}
\text{cd}(M_a^+) &= L(a^+) \\
\text{cd}(M_a^+ \cap M_b^-) &= L(a^+, b^-) \\
\text{cd}(M_a^+ \cap M_b^- \cap M_c^+) &= L(a^+, b^-, c^+). \\
\end{align*}
\]

We can now prove

\[
\begin{align*}
(1) \quad L(a^+, c^+) &= L(a^+, b^+, c^+) + L(a^+, b^-, c^+) \\
(2) \quad L(a^+, b^+) &= L(a^+, b^+, c^+) + L(a^+, b^-, c^+) \\
(3) \quad L(b^-, c^+) &= L(a^+, b^-, c^+) + L(a^+, b^-, c^+) \\
\end{align*}
\]
as follows.

By assumption \( R \) (or Corollary 10.16) we have

\[ U = M_B^+ \cup M_B^- \]

Then by Assumption \( S \) and the algebra of sets,

\[
(4) \quad M_a^+ \cap M_C^+ = (M_a^+ \cap M_C^+) \cap (M_B^+ \cup M_B^-) = (M_a^+ \cap M_B^+) \cup (M_a^+ \cap M_B^- \cap M_C^+) \\
\]

Since no photon can elicit two different instrument reactions at the same measurement occasion,

\[ M_B^+ \cap M_B^- = \emptyset \]

Hence

\[
(5) \quad (M_a^+ \cap M_B^+ \cap M_C^+) \cap (M_a^+ \cap M_B^- \cap M_C^+) = \emptyset \\
\]

From (4) and (5) follows

\[
(6) \quad \text{cd}(M_a^+ \cap M_C^+) = \text{cd}(M_a^+ \cap M_B^+ \cap M_C^+) + \text{cd}(M_a^+ \cap M_B^- \cap M_C^+) \\
\]

(6) is equivalent to equation (1). Equations (2) and (3) are proved similarly.

But equations (1), (2), (3) are identical with equations (1), (2), (3), respectively, of § 4.9. Now lemmas 4.9 and 4.11 can be proved exactly as in Section 4, and Bell’s inequality follows.

Therefore one of the three assumptions \( S, L, \) or \( R \) must be false. It is clear that it is the unrestrained use of the intersection operation which is the villain. Though each of \( M_a^+, M_B^-, M_C^+ \) is defined separately, \( M_a^+ \cap M_B^- \cap M_C^+ \) is not defined, since otherwise we must be able to assign a definite polarization component to each photon in three directions simultaneously.

Collections of elementary particles are thus not Cantorian sets. It is worth noting that we have used only finite collections of elementary particles. The result obtained here has nothing to do with the traditional paradoxes of infinite sets.

11.27 It has been proposed before that \( QM \) demands a logic different from the classical Boolean. The best known such proposal is Birkhoff-von Neumann’s quantum logic (BNQL). It may be worth comparing their ideas with the ideas outlined above.

The atomic sentences of BNQL have the form

\[ (1) \quad \text{At time } t \text{ the value of the dynamic variable } v \text{ for the system } Q \text{ lies in the range } R. \]

From these atomic sentences, molecular sentences are built by means of connectives. The meanings of these connectives are defined on the lattice structure of a Hilbert space.

An example of an atomic sentence of BNQL is

\[ (2) \quad \text{At time } t \text{ the value of the spin component } S_b \text{ for the proton } p \text{ is } +1. \]

But sentence (2) was analyzed in Section 10 as

\[ (2') \quad a_t(p) \Rightarrow a_t^+(p) \]

Birkhoff-von Neumann’s ideas and mine thus seem to be wholly unrelated. The logical structure of BNQL arises when we combine sentences of the form (1). The logical properties we have discovered arise when we analyze further the logical form of the sentences (1).

11.28 During a seminar Stig Kanger raised the following objection to my ideas on Bell’s theorem: "I don’t like the idea. If you change the logic, everything [i.e., QM] becomes even more opaque than it already is."

What I am pressing for is a better understanding of the nature and foundations of logic. To be able to make inferences is to know the general principles for the working of systems. It may well be that these principles vary from one type of systems to another. E.g., we may be forced to accept
the indeterminacy principles as part of the logic of quantum systems.

The point of view of classical formal logic is different. To be able to make inferences is to know the truth conditions of certain sentences. But the formal connectives are also based on the principles for the working of systems, albeit very special systems: systems built out of truth values. How this works out for Boolean logic is indicated in Hansen (1986).

This trick of considering only special systems has some advantages. Logic becomes simple, and it becomes possible to automatize logic. But it also has disadvantages: (i) The implication paradoxes and the entailment problem. (ii) The inability to give an adequate analysis of dispositional terms. (iii) The inability to handle problem fields like, e.g., intuitionistic mathematics, which demand other logical principles. (iv) I believe that it is impossible on the basis of traditional formal logic to give an adequate treatment of the logical problems in QM revealed by Bell's theorem.

A revision of the foundations of logic will not make QM more opaque. It will help to clarify QM instead.

11.29 Mainzer (1987) asks: "What is the price of realism in the quantum world?"

Locality implies realism. This in turn demands a revision of the foundations of logic. But since the presently accepted foundations of logic are shaky and unsatisfactory, such a revision is an improvement and no price at all.

11.30 Is it not so, in spite of all, that the correlation coefficients derived from QM in Section 6 show that there is correlation between spin or polarization components of the twin particles which can only be explained as correlation-at-a-distance?

Wrong! If only two of three variables conjugate in pairs can have values simultaneously, it is impossible to derive Bell's inequality. Without Bell's inequality we have nothing to compare QM with. It becomes impossible to discover any special correlation effects in QM.

What Bell's theorem shows is instead that there must be some very fascinating theorems on the relationship between reality and the universe waiting to be discovered.
NOTES

Section 1. The way of schematically representing experimental arrangements used in this essay is inspired by Rae (1986) and Redhead (1987).

Section 2. Good expositions of the theorems by von Neumann, Jauch and Piron, and Gleason can be found in Bell (1966) and in Jammer (1974). Jammer (1974) also contains a comprehensive exposition of the prehistory of Bell's theorem.

Section 3. The exposition of the EPR argument follows, with one exception, fairly closely the original paper by Einstein, Podolsky, and Rosen (1935). The exception is that we have used spin in two different directions rather than position and momentum for pairs of particles. The idea to simplify the EPR argument in this way is due to D. Bohm. The exposition in Section 3 is also influenced by Redhead (1987).

Section 4. Bell's theorem was first proved by Bell (1964). The basic idea in the proof given here is due to Wigner (1970). The proof is, in expository style, somewhat influenced by d'Espagnat (1979).

Section 5. The proof of the generalized Bell theorem is adapted from d'Espagnat (1984).

Section 6. The proof that the Bell inequality of Section 4 is incompatible with QM is adapted from Rae (1986). The proof that the generalized Bell inequality of Section 5 is incompatible with QM is adapted from d'Espagnat (1984).

Section 8. The exposition in §§ 8.4-8.6 of the quantum potential approach to QM is based on Bohm, Dewdney, and Hilley (1985) and on Rae (1986).

Section 9. The main sources for the exposition of the Copenhagen interpretation have been Bohr (1935) and d'Espagnat (1975). Jammer (1974) and Rae (1986) have also been helpful.
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